OPTIMAL PERTURBATION BOUNDS FOR THE HERMITIAN EIGENVALUE PROBLEM

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Abstract. There is now a large literature on structured perturbation bounds for eigenvalue problems of the form

$$Hx = \lambda Mx,$$

where H and M are Hermitian. These results give relative error bounds on the i^{th} eigenvalue, λ_i, of the form

$$\frac{|\lambda_i - \tilde{\lambda}_i|}{|\lambda_i|},$$

and bound the error in the i^{th} eigenvector in terms of the relative gap,

$$\min_{j \neq i} \frac{|\lambda_i - \lambda_j|}{|\lambda_i|^{1/2}|\lambda_j|^{1/2}}.$$

In general, this theory usually restricts H to be nonsingular and M to be positive definite.

We relax this restriction by allowing H to be singular. For our results on eigenvalues we allow M to be positive semi–definite and for few results we allow it to be more general. For these problems, for eigenvalues that are not zero or infinity under perturbation, it is possible to obtain local relative error bounds. Thus, a wider class of problems may be characterized by this theory.

The theory is applied to the SVD and some of its generalizations. In fact, for structured perturbations, our bound on generalized Hermitian eigenproblems are based upon our bounds for generalized singular value problems.

Although it is impossible to give meaningful relative error bounds on eigenvalues that are not bounded away from zero, we show that the error in the subspace associated with those eigenvalues can be characterized meaningfully.

1. Introduction.

We consider the eigenvalue problem

$$Hx = \lambda Mx, \quad H, M \in \mathbf{C}^{n\times n}, \quad x \in \mathbf{C}^n, \quad \lambda \in \mathbf{C}, \tag{1.1}$$

where H and M are Hermitian matrices. We assume that there exists a nonsingular matrix $X \in \mathbf{C}^{n\times n}$ such that

$$X^*HX = \Omega, \quad X^*MX = J, \tag{1.2}$$

where

$$\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_n), \quad J = \mathrm{diag}(j_1, \ldots, j_n),$$

and

$$\omega_i \in \mathbf{R}, \quad j_i \in \{e^{i\theta} : \theta \in [0, 2\pi]\} \cup \{0\},\ i = 1, 2, \ldots, n.$$

If we restrict X to be unitary, and J to be nonsingular, then $A = HM^{-1}$ is a normal matrix, and (1.1) is a statement of eigenproblem for A.

If we impose the restriction

$$\omega_i \in \mathbf{R}, \quad j_i \in \{0, 1\},$$

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then M is positive semi-definite, and (1.1) is the generalized Hermitian eigenvalue problem. Most of the results in this paper concern this class of eigenvalue problems.

We compare (1.1) to the perturbed problem

$$(H + \Delta H)\tilde{x} = \tilde{\lambda}(M + \Delta M)\tilde{x} \tag{1.3}$$

where ΔH and ΔM are Hermitian and satisfy

$$\|\Delta H\| \leq \delta_H, \quad \|\Delta M\| \leq \delta_M,$$

and δ_H and δ_M are "small" positive real numbers. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of the pencil (1.1) and let $\tilde{\lambda}_1 \geq \ldots \geq \tilde{\lambda}_n$ be eigenvalues of the perturbed pencil (1.3). Starting from the theory of Kato [10], we obtain meaningful bounds on

$$\frac{|\lambda_i - \tilde{\lambda}_i|}{|\lambda_i|}. \tag{1.4}$$

Moreover, for the case when M is positive definite, we give conditions under which we can bound the error in the subspaces in terms of a generalization of the relative gap

$$relgap(\lambda_i) = \min_{j \neq i} \frac{|\lambda_i - \lambda_j|}{|\lambda_i|^{\frac{1}{2}} |\lambda_j|^{\frac{1}{2}}}.$$

This theory generalizes that in papers by Barlow and Demmel [1], Demmel and Veselić [3], Veselić and Slapničar [16], Gu and Eisenstat [9], Li [11, 12], Zha [19], and Eisenstat and Ipsen [4].

We make the following improvements to the theory given in the above papers:

- The bounds on eigenvalues allow for H and M to be singular. These bounds are used to obtain bounds on the singular value decomposition (SVD), the quotient and product SVD for pairs of matrices, and the restricted SVD (RSVD) for matrix triplets.
- The bounds on eigenvectors include bounds on the error in the subspace associated with eigenvalues that are not bounded away from zero.
- The bounds given are local in the sense that each eigenvalue has its own condition number.
- The bounds given are optimal and show clearly the role of structured perturbations.

In §2, we give two simple bounds for the relative error of the form (1.4) under weaker assumptions than have been given in previous works [1, 16, 9, 8] and show how this theory can be applied to the SVD, the quotient and product SVD, and RSVD. In §3, we show how this theory accounts for the effect of structured perturbation on the problem (1.1). In §4, we give bounds on error in subspaces for scaled perturbations. Some examples are given in §5 and our conclusions are in §6.

2. Locally Optimal Perturbation Bounds on Hermitian Pencils.

In this section we first give local condition numbers for eigenvalues. Also, we specialize this result to the cases when M from (1.1) is positive semi-definite and perturbed through factors, and when $M = I$ and H is an indefinite matrix given in factorized form and perturbed through factors. We then derive the perturbation bounds for the singular value decomposition and its generalizations.

2.1. Local Condition Numbers of Eigenvalues.

Consider the perturbed matrix

$$H + \Delta H = H + \delta_H E_H,$$

where $\delta_H = \|\Delta H\|$ and $E_H = \Delta H / \delta_H$. Thus $\|E_H\| = 1$. Let

$$H(\zeta) = H + \zeta E_H, \quad \zeta \in [0, \delta_H]. \tag{2.1}$$

Now consider the family of generalized eigenproblems

$$H(\zeta)x(\zeta) = \lambda(\zeta)Mx(\zeta), \quad \zeta \in [0, \delta_H]. \tag{2.2}$$

where M is fixed. We assume that (1.2) holds for each $\zeta \in [0, \delta_H]$ and some $X(\zeta)$, $\Omega(\zeta)$, and $J(\zeta)$. Let $(\lambda_i(\zeta), x_i(\zeta))$ be the i^{th} eigenpair of (2.2). Define $\mathcal{S}_H(\delta_H)$ to be the set of indices given by

$$\mathcal{S}_H(\delta_H) = \{i : H(\zeta)x_i(\zeta) \neq 0, Mx_i(\zeta) \neq 0 \text{ for all } \zeta \in [0, \delta_H]\}. \tag{2.3}$$

The set $\mathcal{S}_H(\delta_H)$ is the set of eigenvalues for which relative error bounds can be found. The next theorem gives such a bound. Its proof follows that of Theorem 4 in [1, p.773].

THEOREM 2.1. *Let $(\lambda_i(\zeta), x_i(\zeta))$ be the i^{th} eigenpair of the Hermitian pencil in (2.2). Let $\mathcal{S}_H(\delta_H)$ be defined by (2.3). If $i \in \mathcal{S}_H(\delta_H)$, then*

$$\exp(-\delta_H \kappa_i^H) \leq \frac{\lambda_i(\delta_H)}{\lambda_i(0)} \leq \exp(\delta_H \kappa_i^H), \tag{2.4}$$

where

$$\kappa_i^H = \max_{\zeta \in [0,\delta_H]} \frac{|x_i^*(\zeta) E_H x_i(\zeta)|}{|x_i^*(\zeta) H(\zeta) x_i(\zeta)|}.$$

Proof. Assume without loss of generality that $\lambda_i(\zeta) > 0, \zeta \in [0, \delta_H]$, otherwise multiply $H(\zeta)$ from the pencil (2.2) by -1. Now assume that $\lambda_i(\zeta)$ is simple at the point ζ. Then from the classical eigenvalue perturbation theory, since $Mx_i(\zeta) \neq 0$, for sufficiently small ξ we have

$$\lambda_i(\zeta + \xi) = \lambda_i(\zeta) + \xi \frac{x_i^*(\zeta) E_H x_i(\zeta)}{x_i^*(\zeta) M x_i(\zeta)} + O(\xi^2).$$

Since $\lambda_i(\zeta) > 0$ for all ζ, we have

$$\begin{aligned} \frac{\lambda_i(\zeta + \xi)}{\lambda_i(\zeta)} &= 1 + \xi \frac{x_i^*(\zeta) E_H x_i(\zeta)}{\lambda_i(\zeta) x_i^*(\zeta) M x_i(\zeta)} + O(\xi^2) \\ &= 1 + \xi \frac{x_i^*(\zeta) E_H x_i(\zeta)}{x_i^*(\zeta) H(\zeta) x_i(\zeta)} + O(\xi^2). \end{aligned}$$

Thus, we have

$$\left| \frac{d(\ln\ \lambda_i(\zeta))}{d\zeta} \right| \leq \frac{|x_i^*(\zeta) E_H x_i(\zeta)|}{|x_i^*(\zeta) H(\zeta) x_i(\zeta)|}. \tag{2.5}$$

If $\lambda_i(\zeta)$ is simple for all $\zeta \in [0, \delta_H]$, then the bound (2.5) follows by integrating from 0 to δ_H. In Kato [10, Theorem II.6.1,p.139], it is shown that the eigenvalues of $H(\zeta)$ in $\mathcal{S}_H(\delta_H)$ are real analytic, even when they are multiple. Moreover, Kato [10, p.143] goes on to point out that there are only a finite number of ζ where $\lambda_i(\zeta)$ is multiple, so that $\lambda_i(\zeta)$ is continuous and piecewise analytic throughout the interval $[0, \delta_H]$. Thus we can obtain (2.4) by integrating over each of the intervals in which $\lambda_i(\zeta)$ is analytic. ☐

The above bound is "tight" in the sense that if the computable first order approximation

$$\hat{\kappa}_i = \frac{|x_i^*(0) E_H x_i(0)|}{|x_i^*(0) H(0) x_i(0)|}$$

is large, then the corresponding eigenvalue is sensitive. If the perturbation E_H is not structured in any particular way with regard to $x_i(\zeta)$, then the bound from Theorem 2.1 will not be much better than classical normwise bounds [17, 15, 7].

Now we consider perturbing M. Let

$$M + \Delta M = M + \delta_M E_M$$

where $\delta_M = \|\Delta M\|$ and $E_M = \Delta M/\delta_M$. We then let

$$M(\xi) = M + \xi E_M, \quad \xi \in [0, \delta_M].$$

Now consider the family of generalized eigenproblems

$$\tilde{H}\tilde{x}(\xi) = \tilde{\lambda}(\xi)M(\xi)\tilde{x}(\xi), \quad \xi \in [0, \delta_M] \tag{2.6}$$

where $\tilde{H} = H + \Delta H$ is fixed. Let $(\tilde{\lambda}_i(\xi), \tilde{x}_i(\xi))$ be the i^{th} eigenpair of the pencil (2.6). Define the index set $\mathcal{S}_M(\delta_M)$ by

$$\mathcal{S}_M(\delta_M) = \{i : \tilde{H}\tilde{x}_i(\xi) \neq 0, M(\xi)\tilde{x}_i(\xi) \neq 0 \text{ for all } \xi \in [0, \delta_M]\}. \tag{2.7}$$

Now we have an analogous theorem for M.

THEOREM 2.2. *Let $(\tilde{\lambda}_i(\xi), \tilde{x}_i(\xi))$ be the i^{th} eigenpair of the Hermitian pencil in (2.6). Let the index set $\mathcal{S}_M(\delta_M)$ be defined by (2.7). For all $i \in \mathcal{S}_M(\delta_M)$ we have*

$$\exp(-\delta_M \kappa_i^M) \leq \frac{\tilde{\lambda}_i(\delta_M)}{\tilde{\lambda}_i(0)} \leq \exp(\delta_M \kappa_i^M), \tag{2.8}$$

where

$$\kappa_i^M = \max_{\xi \in [0,\delta_M]} \frac{|\tilde{x}_i^*(\xi)E_M\tilde{x}_i(\xi)|}{|\tilde{x}_i^*(\xi)M(\xi)\tilde{x}_i(\xi)|}.$$

Proof. For $i \in \mathcal{S}_M(\delta_M)$ we note that $\tilde{\lambda}_i(\xi) \neq 0$. Thus, let $\tau_i(\xi) = \tilde{\lambda}_i(\xi)^{-1}$, where $\tau_i(\xi)$ is an eigenvalue of

$$M(\xi)\tilde{x}(\xi) = \tau(\xi)\tilde{H}\tilde{x}(\xi).$$

We note that the proof of Theorem 2.1 goes through if we exchange the roles of H and M, thus

$$\exp(-\delta_M \kappa_i^M) \leq \frac{\tau(\delta_M)}{\tau(0)} \leq \exp(\delta_M \kappa_i^M).$$

Taking the reciprocal yields (2.8). □

Now we can consider the general problem (1.3). The following corollary is obvious from Theorems 2.1 and 2.2.

COROLLARY 2.3. *Assume the hypotheses and terminology of Theorems 2.1 and 2.2. Let λ_i, $i = 1, 2, \ldots, n$, be the eigenvalues of the pencil in (1.1) and let $\tilde{\lambda}_i$, $i = 1, 2, \ldots, n$, be the eigenvalues of the pencil in (1.3). Let $\mathcal{S}_{H,M}(\delta_H, \delta_M) = \mathcal{S}_H(\delta_H) \cap \mathcal{S}_M(\delta_M)$. For $i \in \mathcal{S}_{H,M}(\delta_H, \delta_M)$ we have*

$$\exp(-\delta_H \kappa_i^H - \delta_M \kappa_i^M) \leq \frac{\tilde{\lambda}_i}{\lambda_i} \leq \exp(\delta_H \kappa_i^H + \delta_M \kappa_i^M).$$

2.1.1. Semi-definite M.

We now assume that M is a positive semi-definite matrix written in the form

$$M = G^*G, \tag{2.9}$$

where $G \in \mathbf{C}^{m \times n}$. We assume no relationship between m and n. Also, let the perturbation to M be structured according to

$$M + \Delta M = (G + \Delta G)^*(G + \Delta G), \qquad \|\Delta G\| = \delta_G.$$

Again define

$$E_G = \Delta G / \delta_G. \tag{2.10}$$

We now define

$$M(\xi) = G(\xi)^* G(\xi), \qquad G(\xi) = G + \xi E_G.$$

The structure of this perturbation is different from before since

$$M(\xi) = M + \xi(G^* E_G + E_G^* G) + \xi^2 E_G^* E_G, \qquad \xi \in [0, \delta_G] \tag{2.11}$$

is now a quadratic function in ξ instead of a linear one. This leads to the following theorem.

THEOREM 2.4. *Let M be an $n \times n$ positive semi-definite matrix and let G be an $m \times n$ matrix satisfying (2.9). Let $M(\xi)$ be defined by (2.11), let δ_G and E_G be defined by (2.10), and let $(\lambda_i(\xi), x_i(\xi))$ be the i^{th} eigenpair of the pencil $H - \lambda M(\xi)$. Define*

$$\mathcal{S}_G(\delta_G) = \{i : Hx_i(\xi) \neq 0, M(\xi)x_i(\xi) \neq 0, \quad \forall \xi \in [0, \delta_G]\}.$$

Also define $y_i(\xi) = G(\xi)x_i(\xi)/\|G(\xi)x_i(\xi)\|$. For $i \in \mathcal{S}_G(\delta_G)$ we have

$$\exp(-2\delta_G \kappa_i^G) \leq \frac{\lambda_i(\delta_G)}{\lambda_i(0)} \leq \exp(2\delta_G \kappa_i^G), \tag{2.12}$$

where

$$\kappa_i^G = \max_{\xi \in [0, \delta_G]} \frac{|\mathrm{Re}(y_i^*(\xi) E_G x_i(\xi))|}{|y_i^*(\xi) G(\xi) x_i(\xi)|}.$$

Proof. For $i \in \mathcal{S}_G(\delta_G)$ we assume again that $\lambda_i(\xi)$ simple at the point ξ and non-negative for $\xi \in [0, \delta_G]$. Then from classical eigenvalue perturbation theory for sufficiently small ζ we have

$$\lambda_i(\xi + \zeta) = \lambda_i(\xi) - \zeta \lambda_i(\xi) \frac{x_i^*(\xi)[G(\xi)^* E_G + E_G^* G(\xi)] x_i(\xi)}{x_i^*(\xi) M(\xi) x_i(\zeta)} + O(\zeta^2).$$

Using the definitions of $G(\xi)$ and $y_i(\xi)$ the above expression becomes

$$\begin{aligned} \lambda_i(\xi + \zeta) &= \lambda_i(\xi) - \zeta \lambda_i(\xi) \frac{y_i^*(\xi) E_G x_i(\xi) + x_i^*(\xi) E_G^* y_i(\xi)}{y_i^*(\xi) G(\xi) x_i(\xi)} + O(\zeta^2) \\ &= \lambda_i(\xi) - \zeta \lambda_i(\xi) \frac{2\mathrm{Re}(y_i^*(\xi) E_G x_i(\xi))}{y_i^*(\xi) G(\xi) x_i(\xi)} + O(\zeta^2). \end{aligned}$$

Thus,

$$\frac{\lambda_i(\xi + \zeta)}{\lambda_i(\xi)} = 1 - \zeta \frac{2\mathrm{Re}(y_i^*(\xi) E_G x_i(\xi))}{y_i^*(\xi) G(\xi) x_i(\xi)} + O(\zeta^2),$$

and we have

$$\left| \frac{d(\ln\ \lambda_i(\xi))}{d\xi} \right| \leq \frac{2|\mathrm{Re}(y_i^*(\xi) E_G x_i(\xi))|}{|y_i^*(\xi) G(\xi) x_i(\xi)|} \leq 2\kappa_i^G.$$

If $\lambda_i(\xi)$ is simple for all $\xi \in [0, \delta_G]$, then the bound (2.12) follows by integrating from 0 to δ_G. Otherwise, the argument given in the proof of Theorem 2.1 applies to obtain (2.12). □

An analogy of Corollary 2.3 can be obtained by combining Theorems 2.1 and 2.4.

2.1.2. Hermitian Matrices Perturbed Through Factors. A corollary is related to a result for perturbation by factors due to Veselić and Slapničar [16].

COROLLARY 2.5. *Let* $H \in \mathbf{C}^{n\times n}$ *have the form* $H = GJG^*$, *where* $G \in \mathbf{C}^{n\times m}$ *and* $J \in \mathbf{C}^{m\times m}$ *is normal and nonsingular. Let* $\Delta G = \delta_G E_G \in \mathbf{C}^{n\times m}$, *where* $\|E_G\| = 1$. *Define* $G(\xi) = G + \xi E_G$ *for* $\xi \in [0, \delta_G]$ *and let* $H(\xi) = G(\xi)JG^*(\xi)$. *Let* $(\lambda_i(\xi), x_i(\xi))$ *be the* i^{th} *eigenpair of* $H(\xi)$ *and let* i *be an index such that* $\lambda_i(\xi) \neq 0$ *for* $\xi \in [0, \delta_G]$. *Also define* $y_i(\xi)$ *by*

$$y_i(\xi) = \frac{JG^*(\xi)x_i(\xi)}{\|JG^*(\xi)x_i(\xi)\|}. \tag{2.13}$$

Then $\lambda_i(\delta_G)$ *satisfies*

$$\exp(-2\delta_G \kappa_i^G) \le \frac{\lambda_i(\delta_G)}{\lambda_i(0)} \le \exp(2\delta_G \kappa_i^G), \tag{2.14}$$

where

$$\kappa_i^G = \max_{\xi\in[0,\delta_G]} \frac{|\mathrm{Re}(y_i^*(\xi)E_G^* x_i(\xi))|}{|y_i^*(\xi)G^*(\xi)x_i(\xi)|}. \tag{2.15}$$

Proof. The key observation is to recognize that $y_i(\xi)$ in (2.13) satisfies

$$G^*(\xi)G(\xi)y_i(\xi) = \lambda_i(\xi)J^{-1}y_i(\xi), \quad \xi \in [0, \delta_G].$$

We note that this is a pencil of the form given in Theorem 2.4. We then note that, since $x_i(\xi)$ is an eigenvector of $H(\xi) = G(\xi)JG^*(\xi)$, we have

$$x_i(\xi) = \frac{G(\xi)y_i(\xi)}{\|G(\xi)y_i(\xi)\|} = \frac{G(\xi)JG^*(\xi)x_i(\xi)}{\|G(\xi)JG^*(\xi)x_i(\xi)\|}.$$

Thus, the conclusion of Theorem 2.4 gives the bound (2.14) with κ_i^G given by (2.15). □

2.2. The Singular Value Decomposition and Generalizations. In this section we describe the singular value decomposition and its generalizations, and apply the perturbation results from §2 to these decompositions.

2.2.1. The Decompositions. The singular value decomposition (SVD) of a matrix A, the generalized quotient SVD (QSVD) and product SVD (PSVD) of a matrix pair (A, B), and the restricted singular value decomposition(RSVD) of a matrix triplet (A, B, C) are all special Hermitian generalized eigenvalue problems.

The SVD of a matrix $A \in \mathbf{C}^{m\times n}$ is given by

$$A = U\Sigma V^*, \quad \Sigma = \begin{array}{c} \\ k \\ m-k \end{array} \!\!\begin{array}{c} \begin{array}{cc} k & n-k \end{array} \\ \begin{pmatrix} \Psi & 0 \\ 0 & 0 \end{pmatrix} \end{array}, \quad k = \mathrm{rank}(A),$$

where $U \in \mathbf{C}^{m\times m}$ and $V \in \mathbf{C}^{n\times n}$ are unitary, and

$$\Psi = \mathrm{diag}(\sigma_1, \dots, \sigma_k) \in \mathbf{R}^{m\times n}$$

is nonnegative.

From Van Loan [13] and Paige and Saunders [14], the generalized quotient SVD (QSVD) of the matrix pair (A, B), $A \in \mathbf{C}^{m\times n}$, $B \in \mathbf{C}^{p\times n}$ is given by

$$A = U\Sigma_A X^{-1}, \quad \Sigma_A = \begin{array}{c} \\ s_1 \\ s_2 \\ t_1 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \end{array} \\ \begin{pmatrix} \Psi_A & 0 & 0 & 0 \\ 0 & I_{s_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \quad s_1 + s_2 = \mathrm{rank}(A),$$

$$B = V\Sigma_B X^{-1}, \quad \Sigma_B = \begin{array}{c c} & \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ t_2 \end{array} & \begin{pmatrix} \Psi_B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{s_3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \quad s_1 + s_3 = \mathrm{rank}(B),$$

$$\Psi_A = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_{s_1}), \quad \Psi_B = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_{s_1}),$$

where $U \in \mathbf{C}^{m\times m}$ and $V \in \mathbf{C}^{p\times p}$ are unitary and $X \in \mathbf{C}^{n\times n}$ is nonsingular. We can make the regularity assumptions

$$\|Xe_i\| = 1, \quad \|X^{-1}\| = \| \begin{pmatrix} A \\ B \end{pmatrix} \|, \quad \alpha_i^2 + \beta_i^2 = 1, \quad i = 1, 2, \ldots, s_1.$$

If B is square and nonsingular this yields the SVD of AB^{-1}, hence the name *quotient* SVD.

Also of significance is the product SVD (PSVD) of the matrix pair (A, B) due to Hammarling and Fernando [6]. It gives a form for the singular value decomposition of AB, where $A \in \mathbf{C}^{m\times n}$ and $B \in \mathbf{C}^{n\times p}$, can be written as

$$A = U\Sigma_A X^{-1}, \quad \Sigma_A = \begin{array}{c c} & \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \\ t_1 \end{array} & \begin{pmatrix} \Psi_A & 0 & 0 & 0 \\ 0 & I_{s_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \quad s_1 + s_2 = \mathrm{rank}(A),$$

$$B = X\Sigma_B V^*, \quad \Sigma_B = \begin{array}{c c} & \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ t_2 \end{array} & \begin{pmatrix} \Psi_B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{s_3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \quad s_1 + s_3 = \mathrm{rank}(B),$$

where again U and V are unitary, X is nonsingular, and $\Sigma_A\Sigma_B$ is the diagonal matrix of the singular values of AB.

Rather than express the perturbation theory of the QSVD and PSVD separately, we instead give the restricted SVD of a matrix triplet (A, B, C) due to Zha [18] which includes the QSVD and PSVD as special cases. It has the form

$$A = Y^{-*}\Sigma_A X^{-1} \in \mathbf{C}^{m\times n},$$

$$B = V\Sigma_B X^{-1} \in \mathbf{C}^{p\times n},$$

$$C = Y^{-*}\Sigma_C U^* \in \mathbf{C}^{m\times q},$$

where U and V are unitary, X and Y are nonsingular, and Σ_A, Σ_B and Σ_C are nonnegative and diagonal. If B and C are nonsingular, this implicitly gives the SVD of $C^{-1}AB^{-1}$. If $C = I$, then this is the QSVD of (A, B), and if $A = I$, then this is the PSVD of (B, C). The matrices Σ_A, Σ_B

and Σ_C can be written in the following form

$$
\left(\begin{array}{c|c} \Sigma_A & \Sigma_B \\ \hline \Sigma_C & 0 \end{array}\right) =
\begin{array}{c}
\\ q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ r_1 \\ r_2 \\ r_3 \\ r_4
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & t_1 & t_2 & t_3 & t_4 \end{array} \\
\left(\begin{array}{cccccccccc}
I & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Psi_A & 0 & 0 & 0 & 0 & \Psi_B & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & & & & \\
0 & 0 & I & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & \Psi_C & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & I & 0 & & & &
\end{array}\right)
\end{array}
$$

where

$$s_1 = q_1 = t_1 = |\{i : Ax_i, Bx_i \neq 0, C^* y_i = 0\}|,$$

$$s_2 = q_2 = |\{i : Ax_i \neq 0, Bx_i = C^* y_i = 0\}|,$$

$$s_3 = q_3 = |\{i : Ax_i \neq 0, Bx_i = 0, C^* y_i \neq 0\}|,$$

$$s_4 = q_4 = t_3 = r_3 = |\{i : Ax_i, Bx_i, C^* y_i \neq 0\}|,$$

$$s_5 = r_4 = |\{i : Ax_i = 0, C^* y_i \neq 0\}|, \quad q_5 = t_4 = |\{i : A^* y_i = 0, Bx_i \neq 0\}|,$$

$$s_6 = |\{i : Ax_i = 0, C^* y_i = 0\}|, \quad q_6 = |\{i : A^* y_i = 0, Bx_i = 0\}|,$$

$$t_2 = |\{i : B^* v_i = 0, A^* y_i \neq 0\}|,$$

$$r_2 = |\{i : Cu_i = 0, Ay_i \neq 0\}|.$$

Here $k = q_4 = s_4 = t_3 = r_3$ is the dimension of the set of "interesting" singular values of (A, B, C).

2.2.2. The Perturbation Theory. Again from Zha [19, 18], the RSVD is equivalent to the matrix pencil

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \lambda \begin{pmatrix} B^* B & 0 \\ 0 & CC^* \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}. \tag{2.16}$$

Here each singular value triplet $(\alpha_i, \beta_i, \gamma_i)$ such that $\alpha_i \beta_i \gamma_i \neq 0$ corresponds to two eigenvalues of the pencil (2.16). They are

$$\lambda_i = \frac{\alpha_i}{\beta_i \gamma_i}, \quad \lambda_{i+k} = -\frac{\alpha_i}{\beta_i \gamma_i}, \tag{2.17}$$

and they correspond to the eigenvectors

$$z_i = \frac{1}{\sqrt{\beta_i^2 + \gamma_i^2}} \begin{pmatrix} \gamma_i x_i \\ \beta_i y_i \end{pmatrix}, \quad z_{i+k} = \frac{1}{\sqrt{\beta_i^2 + \gamma_i^2}} \begin{pmatrix} \gamma_i x_i \\ -\beta_i y_i \end{pmatrix}.$$

The positive values of λ_i are called the generalized singular values of the matrix triplet (A, B, C). The theory laid out in the previous section will quickly yield perturbation bounds for this eigenvalue problem.

We now compare the generalized singular values of (A, B, C) to those of a "nearby" triplet $(\tilde{A}, \tilde{B}, \tilde{C})$. This generalizes some results in [19] to allow A, B, and C to be rank deficient, and is proven with weaker assumptions. We need to define some terms. Let

$$\Delta A = \tilde{A} - A, \quad \Delta B = \tilde{B} - B, \quad \Delta C = \tilde{C} - C, \tag{2.18}$$

$$\delta_A = \|\Delta A\|, \quad \delta_B = \|\Delta B\|, \quad \delta_C = \|\Delta C\|, \tag{2.19}$$

$$E_A = \Delta A/\delta_A, \quad E_B = \Delta B/\delta_B, \quad E_C = \Delta C/\delta_C. \tag{2.20}$$

We also define

$$\delta_{BC} = \max\{\delta_B, \delta_C\}, \quad F_B = \Delta B/\delta_{BC}, \quad F_C = \Delta C/\delta_{BC}. \tag{2.21}$$

Let

$$A(\zeta) = A + \zeta E_A, \quad \zeta \in [0, \delta_A], \tag{2.22}$$

$$B(\xi) = B + \xi F_B, \quad C(\xi) = C + \xi F_C, \quad \xi \in [0, \delta_{BC}]. \tag{2.23}$$

Note that $\tilde{A} = A(\delta_A)$, $\tilde{B} = B(\delta_{BC})$ and $\tilde{C} = C(\delta_{BC})$. For $\zeta \in [0, \delta_A]$, let the RSVD of the matrix triplet $(A(\zeta), B, C)$ be given by

$$\begin{aligned} A(\zeta) &= Y^{-*}(\zeta)\Sigma_A(\zeta)X^{-1}(\zeta), \quad \Sigma_A(\zeta) = \mathrm{diag}(\alpha_i(\zeta)), \\ B &= V(\zeta)\Sigma_B(\zeta)X^{-1}(\zeta), \quad \Sigma_B = \mathrm{diag}(\beta_i(\zeta)), \\ C &= Y^{-*}(\zeta)\Sigma_C(\zeta)U^*(\zeta), \quad \Sigma_C = \mathrm{diag}(\gamma_i(\zeta)). \end{aligned} \tag{2.24}$$

For appropriate values of i, define

$$x_i(\zeta) = X(\zeta)e_i, \quad y_i(\zeta) = Y(\zeta)e_i, \quad \zeta \in [0, \delta_A]. \tag{2.25}$$

For $\xi \in [0, \delta_{BC}]$ let the RSVD of the matrix triplet $(\tilde{A}, B(\xi), C(\xi))$ be given by

$$\begin{aligned} \tilde{A} &= \tilde{Y}^{-*}(\xi)\tilde{\Sigma}_A(\xi)\tilde{X}^{-1}(\xi), \quad \tilde{\Sigma}_A = \mathrm{diag}(\tilde{\alpha}_i(\xi)), \\ B(\xi) &= \tilde{V}(\xi)\tilde{\Sigma}_B(\xi)\tilde{X}^{-1}(\xi), \quad \tilde{\Sigma}_B = \mathrm{diag}(\tilde{\beta}_i(\xi)), \\ C(\xi) &= \tilde{Y}^{-*}(\xi)\tilde{\Sigma}_C(\xi)\tilde{U}^*(\zeta), \quad \tilde{\Sigma}_C = \mathrm{diag}(\tilde{\gamma}_i(\xi)). \end{aligned} \tag{2.26}$$

For appropriate values of i, define

$$\begin{aligned} \tilde{x}_i(\xi) &= \tilde{X}(\xi)e_i, \quad \tilde{y}_i(\zeta) = \tilde{Y}(\xi)e_i, \quad \xi \in [0, \delta_{BC}], \\ \tilde{u}_i(\xi) &= \tilde{U}(\xi)e_i, \quad \tilde{v}_i(\zeta) = \tilde{V}(\xi)e_i, \quad \xi \in [0, \delta_{BC}]. \end{aligned} \tag{2.27}$$

Finally, define the set of indices

$$\mathcal{S}_{ABC}(\delta_A, \delta_{BC}) = \mathcal{S}_A(\delta_A) \cap \mathcal{S}_{BC}(\delta_{BC}), \tag{2.28}$$

where

$$\mathcal{S}_A(\delta_A) = \{i : \alpha_i(\zeta)\beta_i(\zeta)\gamma_i(\zeta) \neq 0, \text{ for all } \zeta \in [0, \delta_A]\},$$

$$\mathcal{S}_{BC}(\delta_{BC}) = \{i : \tilde{\alpha}_i(\xi)\tilde{\beta}_i(\xi)\tilde{\gamma}_i(\xi) \neq 0, \text{ for all } \xi \in [0, \delta_{BC}]\}.$$

We note that if $i \in \mathcal{S}_{ABC}(\delta_A, \delta_{BC})$, then

$$\lambda_i = \frac{\alpha_i}{\beta_i\gamma_i}, \quad \tilde{\lambda}_i = \frac{\tilde{\alpha}_i}{\tilde{\beta}_i\tilde{\gamma}_i}, \tag{2.29}$$

where $(\alpha_i, \beta_i, \gamma_i)$ and $(\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\gamma}_i)$ are singular triplets of (A, B, C) and $(\tilde{A}, \tilde{B}, \tilde{C})$. For such i, it is meaningful to bound $\tilde{\lambda}_i/\lambda_i$.

THEOREM 2.6. *Let (A, B, C) and $(\tilde{A}, \tilde{B}, \tilde{C})$ be matrix triplets such that $A, \tilde{A} \in \mathbf{C}^{m\times n}$, $B, \tilde{B} \in \mathbf{C}^{p\times n}$, $C, \tilde{C} \in \mathbf{C}^{q\times n}$. Let $\delta_A = \|\tilde{A} - A\|$, $\delta_B = \|\tilde{B} - B\|$, and $\delta_C = \|\tilde{C} - C\|$, and let δ_{BC} be given by (2.21). Let E_A, E_B and E_C be given by (2.20), and let $\mathcal{S}_{ABC}(\delta_A, \delta_{BC})$ be given by (2.28). Let $x_i(\zeta), y_i(\zeta), \zeta \in [0, \delta_A]$ be given by (2.25) and let $\tilde{x}_i(\xi), \tilde{y}_i(\xi), \tilde{u}_i(\xi), \tilde{v}_i(\xi), \xi \in [0, \delta_{BC}]$ be given by (2.27). For all $i \in \mathcal{S}_{ABC}(\delta_A, \delta_{BC})$, if λ_i and $\tilde{\lambda}_i$ are given by (2.29), we have*

$$\exp(-\delta_A\kappa_i^A - \delta_B\kappa_i^B - \delta_C\kappa_i^C) \leq \frac{\tilde{\lambda}_i}{\lambda_i} \leq \exp(\delta_A\kappa_i^A + \delta_B\kappa_i^B + \delta_C\kappa_i^C), \tag{2.30}$$

where

$$\kappa_i^A = \max_{\zeta\in[0,\delta_A]} \frac{|\mathrm{Re}(y_i^*(\zeta)E_A x_i(\zeta))|}{|y_i^*(\zeta)A(\zeta)x_i(\zeta)|}, \tag{2.31}$$

$$\kappa_i^B = \max_{\xi\in[0,\delta_{BC}]} \frac{|\mathrm{Re}(\tilde{v}_i^*(\xi)E_B\tilde{x}_i(\xi))|}{|\tilde{v}_i^*(\xi)B(\xi)\tilde{x}_i(\xi)|},$$

$$\kappa_i^C = \max_{\xi\in[0,\delta_{BC}]} \frac{|\mathrm{Re}(\tilde{y}_i^*(\xi)E_C\tilde{u}_i(\xi))|}{|\tilde{y}_i^*(\xi)C(\xi)\tilde{u}_i(\xi)|}.$$

Proof. This theorem is proven by considering the two doubled generalized eigenvalue problems associated with the RSVDs (2.24) and (2.26). The RSVD (2.24) is equivalent to eigenvalue problem for the pair $(H(\zeta), M), \zeta \in [0, \delta_A]$ where

$$H(\zeta) = \begin{pmatrix} 0 & A^* + \zeta E_A^* \\ A + \zeta E_A & 0 \end{pmatrix}, \quad M = \mathrm{diag}(B^*B, CC^*).$$

We note that for $i \in \mathcal{S}_{ABC}(\delta_A, \delta_{BC})$, the eigenvector of $H(\zeta)$ has the form

$$z(\zeta) = (\gamma_i(\zeta)x_i^T(\zeta), \beta_i(\zeta)y_i^T(\zeta))^T.$$

If we let $\lambda_i(\zeta)$ be the i^{th} eigenvalue of the pair $(H(\zeta), M), \zeta \in [0, \delta_A]$, and define

$$E_H = \begin{pmatrix} 0 & E_A^* \\ E_A & 0 \end{pmatrix},$$

then κ_i^H in (2.1) satisfies

$$\begin{aligned}
\kappa_i^H &= \max_{\zeta\in[0,\delta_A]} \frac{|\mathrm{Re}(z_i^*(\zeta)E_H z_i(\zeta))|}{|z_i^*(\zeta)H(\zeta)z_i^*(\zeta)|} \\
&= \max_{\zeta\in[0,\delta_A]} \frac{\beta_i(\zeta)\gamma_i(\zeta)|\mathrm{Re}(y_i^*(\zeta)E_A x_i(\zeta) + x_i^*(\zeta)E_A^* y_i(\zeta))|}{\beta_i(\zeta)\gamma_i(\zeta)|y_i^*(\zeta)A(\zeta)x_i^*(\zeta) + x_i^*(\zeta)A^*(\zeta)y_i(\zeta)|} \\
&= \max_{\zeta\in[0,\delta_A]} \frac{2|\mathrm{Re}(y_i^*(\zeta)E_A x_i(\zeta))|}{2|y_i^*(\zeta)A(\zeta)x_i^*(\zeta)|} = \kappa_i^A
\end{aligned}$$

according to (2.31). Thus $\lambda_i(\delta_A)$ satisfies

$$\exp(-\delta_A \kappa_i^A) \le \frac{\lambda_i(\delta_A)}{\lambda_i(0)} \le \exp(\delta_A \kappa_i^A). \tag{2.32}$$

Now for $i \in \mathcal{S}_{ABC}(\delta_A, \delta_{BC})$, let $\tilde{\lambda}_i(0) = \lambda_i(\delta_A)$, and let $(\tilde{\lambda}_i(\xi), \tilde{z}_i(\xi))$ be the i^{th} eigenpair of the pair $(\tilde{H}, M(\xi))$, where $\tilde{H} = H(\delta_A)$ and

$$M(\xi) = G^*(\xi)G(\xi), \quad G(\xi) = \mathrm{diag}(B, C^*) + \xi\, \mathrm{diag}(\eta_B E_B, \eta_C E_C^*).$$

Here $\eta_B = \delta_B/\delta_{BC}$ and $\eta_C = \delta_C/\delta_{BC}$. We note that

$$s_i(\xi) = \frac{G(\xi)\tilde{z}_i(\xi)}{\|G(\xi)\tilde{z}_i(\xi)\|} = \frac{\tilde{\beta}_i(\xi)\tilde{\gamma}_i(\xi)(\tilde{v}_i^*(\xi), \tilde{u}_i^*(\xi))}{\|\tilde{\beta}_i(\xi)\tilde{\gamma}_i(\xi)(\tilde{v}_i^*(\xi), \tilde{u}_i^*(\xi))^*\|} = \sqrt{0.5}(\tilde{v}_i^*(\xi), \tilde{u}_i^*(\xi))^*,$$

thus we can write κ_i^G from (2.12) as

$$\begin{aligned} \kappa_i^G &= \max_{\xi \in [0, \delta_{BC}]} \frac{2|\mathrm{Re}(s_i^*(\xi)\, \mathrm{diag}(\eta_B E_B, \eta_C E_C^*)\tilde{z}_i(\xi))|}{|s_i^*(\xi)G(\xi)\tilde{z}_i(\xi)|} \\ &= \max_{\xi \in [0, \delta_{BC}]} \frac{2|\mathrm{Re}(\eta_B \tilde{\gamma}_i(\xi)\tilde{v}_i^*(\xi)E_B\tilde{x}_i(\xi) + \eta_C \tilde{\beta}_i(\xi)\tilde{u}_i^*(\xi)E_C\tilde{y}_i(\xi))|}{|\tilde{\gamma}_i(\xi)\tilde{v}_i^*(\xi)B(\xi)\tilde{x}_i(\xi) + \tilde{\beta}_i(\xi)\tilde{y}_i^*(\xi)C(\xi)\tilde{u}_i(\xi)|} \end{aligned}$$

Thus,

$$\begin{aligned} \kappa_i^G \le{} & \max_{\xi \in [0, \delta_{BC}]} \frac{2|\mathrm{Re}(\eta_B \tilde{\gamma}_i(\xi)\tilde{v}_i^*(\xi)E_B\tilde{x}_i(\xi))|}{|\tilde{\gamma}_i(\xi)\tilde{v}_i^*(\xi)B(\xi)\tilde{x}_i(\xi) + \tilde{\beta}_i(\xi)\tilde{y}_i^*(\xi)C(\xi)\tilde{u}_i(\xi)|} \\ & + \max_{\xi \in [0, \delta_{BC}]} \frac{2|\mathrm{Re}(\eta_C \tilde{\beta}_i(\xi)\tilde{v}_i^*(\xi)E_C\tilde{y}_i(\xi))|}{|\tilde{\gamma}_i(\xi)\tilde{v}_i^*(\xi)B(\xi)\tilde{x}_i(\xi) + \tilde{\beta}_i(\xi)\tilde{y}_i^*(\xi)C(\xi)\tilde{u}_i(\xi)|}. \end{aligned} \tag{2.33}$$

The first term in (2.33) is bounded by

$$\begin{aligned} \max_{\xi \in [0, \delta_{BC}]} \frac{2|\mathrm{Re}(\eta_B \tilde{\gamma}_i(\xi)\tilde{v}_i^*(\xi)E_B\tilde{x}_i(\xi))|}{|\tilde{\gamma}_i(\xi)\tilde{\beta}_i(\xi) + \tilde{\beta}_i(\xi)\tilde{\gamma}_i(\xi)|} &= \max_{\xi \in [0, \delta_{BC}]} \frac{2|\mathrm{Re}(\eta_B \tilde{\gamma}_i(\xi)\tilde{v}_i^*(\xi)E_B\tilde{x}_i(\xi))|}{2|\tilde{\gamma}_i(\xi)\tilde{\beta}_i(\xi)|} \\ &= \eta_B \max_{\xi \in [0, \delta_{BC}]} \frac{|\mathrm{Re}(\tilde{v}_i^*(\xi)E_B\tilde{x}_i(\xi))|}{|\tilde{v}_i^*(\xi)B(\xi)x_i(\xi)|} = \eta_B \kappa_i^B. \end{aligned}$$

By a symmetric argument, the second term in (2.33) is bounded by

$$\eta_C \max_{\xi \in [0, \delta_{BC}]} \frac{|\mathrm{Re}(\tilde{v}_i^*(\xi)E_C\tilde{y}_i(\xi))|}{|\tilde{y}_i^*(\xi)C(\xi)\tilde{u}_i(\xi)|} = \eta_C \kappa_i^C.$$

Thus,

$$\kappa_i^G \le \eta_B \kappa_i^B + \eta_C \kappa_i^C. \tag{2.34}$$

Using (2.34) we have

$$\exp(-\delta_{BC}\kappa_i^G) \le \frac{\tilde{\lambda}_i(\delta_{BC})}{\tilde{\lambda}_i(0)} \le \exp(\delta_{BC}\kappa_i^G). \tag{2.35}$$

The combination of (2.34) and (2.35) yields

$$\exp(-\delta_B\kappa_i^B - \delta_C\kappa_i^C) \le \frac{\tilde{\lambda}_i(\delta_{BC})}{\tilde{\lambda}_i(0)} \le \exp(\delta_B\kappa_i^B + \delta_C\kappa_i^C).$$

Using the fact that $\lambda_i(\delta_A) = \tilde{\lambda}_i(0)$ and the bound (2.32) yields (2.30). □

Two corollaries to this theorem yield perturbation bounds for the QSVD and PSVD. For the QSVD consider the matrix pair (A, B) and the "nearby" pair $(\tilde{A}, \tilde{B})$, define ΔA and ΔB as in (2.18), define δ_A and δ_B as in (2.19), and E_A and E_B as in (2.20). We define $A(\zeta)$ as in (2.24) and $B(\xi)$ as in (2.23). For $\zeta \in [0, \delta_A]$, let the QSVD of $(A(\zeta), B)$ be given by

$$A(\zeta) = U(\zeta)\Sigma_A(\zeta)X^{-1}(\zeta), \quad \Sigma_A(\zeta) = \mathrm{diag}(\alpha_i(\zeta)), \tag{2.36}$$

$$B = V(\zeta)\Sigma_B(\zeta)X^{-1}(\zeta), \quad \Sigma_B = \mathrm{diag}(\beta_i(\zeta)),$$

and the QSVD of $(\tilde{A}, B(\xi)), \xi \in [0, \delta_B]$, be given by

$$\tilde{A} = \tilde{Y}^{-*}(\xi)\tilde{\Sigma}_A(\xi)\tilde{X}^{-1}(\xi), \quad \tilde{\Sigma}_A = \mathrm{diag}(\tilde{\alpha}_i(\xi), \tag{2.37}$$

$$B(\xi) = \tilde{V}(\xi)\tilde{\Sigma}_B(\xi)\tilde{X}^{-1}(\xi), \quad \tilde{\Sigma}_B = \mathrm{diag}(\tilde{\beta}_i(\xi)).$$

COROLLARY 2.7. *Let (A, B) and $(\tilde{A}, \tilde{B})$ be matrix pairs such that $A, \tilde{A} \in \mathbf{C}^{m\times n}$ and $B, \tilde{B} \in \mathbf{C}^{p\times n}$. Let δ_A and δ_B be given in (2.19). Let E_A and E_B be given by (2.20). Let $\mathcal{S}_{AB}(\delta_A, \delta_B)$ be given by (2.28) with $\delta_{BC} = \delta_B$ and $\gamma_i(\zeta) = \gamma_i(\xi) = 1$ for all i, ζ, and ξ. Let $x_i(\zeta) = X(\zeta)e_i, u_i(\zeta) = U(\zeta)e_i, \zeta \in [0, \delta_A]$, where $X(\zeta)$ and $U(\zeta)$ are from (2.36), and let $\tilde{x}_i(\xi) = \tilde{X}(\xi)e_i, \tilde{v}_i(\xi) = \tilde{V}(\xi)e_i, \xi \in [0, \delta_B]$, where $\tilde{X}(\xi)$ and $\tilde{V}(\xi)$ are from (2.37). For all $i \in \mathcal{S}_{AB}(\delta_A, \delta_B)$, if $\lambda_i = \alpha_i/\beta_i$ and $\tilde{\lambda}_i = \tilde{\alpha}_i/\tilde{\beta}_i$, where (α_i, β_i) and $(\tilde{\alpha}_i, \tilde{\beta}_i)$, are the corresponding quotient singular value pairs of (A, B) and $(\tilde{A}, \tilde{B})$, respectively, we have*

$$\exp(-\delta_A\kappa_i^A - \delta_B\kappa_i^B) \le \frac{\tilde{\lambda}_i}{\lambda_i} \le \exp(\delta_A\kappa_i^A + \delta_B\kappa_i^B),$$

where

$$\kappa_i^A = \max_{\zeta\in[0,\delta_A]} \frac{|\mathrm{Re}(u_i^*(\zeta)E_A x_i(\zeta))|}{|u_i^*(\zeta)A(\zeta)x_i(\zeta)|},$$

$$\kappa_i^B = \max_{\xi\in[0,\delta_B]} \frac{|\mathrm{Re}(\tilde{v}_i^*(\xi)E_B\tilde{x}_i(\xi))|}{|\tilde{v}_i^*(\xi)B(\xi)\tilde{x}_i(\xi)|}.$$

This is proven from Theorem 2.6 by simply observing the QSVD of (A, B) is the RSVD of (A, B, I) and making appropriate substitutions.

A perturbation bound for the PSVD follows from the following characterization. Consider the matrix pair (A, B) and the "nearby" pair $(\tilde{A}, \tilde{B})$, define ΔA and ΔB as in (2.18), define δ_A and δ_B as in (2.19), and E_A and E_B as in (2.20). We now let

$$\delta_{AB} = \max\{\delta_A, \delta_B\}, \quad F_A = \delta A/\delta_{AB}, \quad F_B = \delta B/\delta_{AB}, \tag{2.38}$$

and let

$$A(\zeta) = A + \zeta F_A, \quad B(\zeta) = B + \zeta F_B.$$

Then the pair $(A(\zeta), B(\zeta))$ has the PSVD given by

$$A(\zeta) = U(\zeta)\Sigma_A(\zeta)X^{-1}(\zeta), \quad \Sigma_A(\zeta) = \mathrm{diag}(\alpha_i(\zeta)), \tag{2.39}$$

$$B(\zeta) = X(\zeta)\Sigma_B(\zeta)V^*(\zeta), \quad \Sigma_B(\zeta) = \mathrm{diag}(\beta_i(\zeta)).$$

We also need to define the set

$$\mathcal{S}_{AB}(\delta_{AB}) = \{i : \alpha_i(\zeta)\beta_i(\zeta) \neq 0, \zeta \in [0, \delta_{AB}]\}. \tag{2.40}$$

We note that we need only one set of perturbations for the PSVD.

COROLLARY 2.8. *Let (A, B) and $(\tilde{A}, \tilde{B})$ be matrix pairs such that $A, \tilde{A} \in \mathbf{C}^{m\times n}$ and $B, \tilde{B} \in \mathbf{C}^{n\times p}$. Let $\delta_A = \|\tilde{A} - A\|$ and $\delta_B = \|\tilde{B} - B\|$. Let δ_{AB} be given in (2.38). Let E_A and E_B be given by (2.20), and let $\mathcal{S}_{AB}(\delta_{AB})$ be given by (2.40). Let $x_i(\zeta) = X(\zeta)^{-*}e_i, u_i(\zeta) = U(\zeta)e_i, \zeta \in [0, \delta_A]$, where $X(\zeta)$ and $U(\zeta)$ are from (2.36), and let $z_i(\zeta) = X^{-1}(\zeta)e_i, v_i(\zeta) = V(\zeta)e_i$, $\zeta \in [0, \delta_{AB}]$, where $\tilde{X}(\zeta)$ and $\tilde{V}(\zeta)$ are from (2.37). For all $i \in \mathcal{S}_{AB}(\delta_{AB})$, if $\lambda_i = \alpha_i\beta_i$ and $\tilde{\lambda}_i = \tilde{\alpha}_i\tilde{\beta}_i$, where (α_i, β_i) and $(\tilde{\alpha}_i, \tilde{\beta}_i)$ are the corresponding product singular value pairs of (A, B) and $(\tilde{A}, \tilde{B})$, respectively, we have*

$$\exp(-\delta_A\kappa_i^A - \delta_B\kappa_i^B) \leq \frac{\tilde{\lambda}_i}{\lambda_i} \leq \exp(\delta_A\kappa_i^A + \delta_B\kappa_i^B),$$

where

$$\kappa_i^A = \max_{\zeta\in[0,\delta_{AB}]} \frac{|\mathrm{Re}(u_i^*(\zeta)E_Ax_i(\zeta)|}{|u_i^*(\zeta)A(\zeta)x_i(\zeta)|},$$

$$\kappa_i^B = \max_{\xi\in[0,\delta_{AB}]} \frac{|\mathrm{Re}(z_i^*(\zeta)E_Bv_i(\zeta)|}{|z_i^*(\zeta)B(\zeta)v_i(\zeta)|}.$$

Proof. From Zha [18], we note that the PSVD of (A, B) is the RSVD of (I, A, B). Thus the PSVD may be written in RSVD form as

$$I = Y^{-*}X^{-1},$$

$$A = U\Sigma_AX^{-1},$$

$$B^* = Y^{-*}\Sigma_BV^*.$$

The first line states that $Y^* = X^{-1}$, thus $B = Y^{-*}\Sigma_BV^*$. Using this characterization with Theorem 2.6 yields the appropriate result. □

We now consider the SVD of $A + \Delta A$. From just considering the RSVD of the triplet (A, I, I) we obtain the following corollary.

COROLLARY 2.9. *Let $A, \Delta A \in \mathbf{C}^{m\times n}, m \geq n$, and let $\delta_A = \|\Delta A\|$ and $E_A = \Delta A/\delta_A$. Define $A(\zeta) = A + \zeta E_A$ for $\zeta \in [0, \delta_A]$. Let $A(\zeta)$ have the singular value decomposition*

$$A(\zeta) = U(\zeta)\Sigma(\zeta)V(\zeta)^*, \quad \zeta \in [0, \delta_A],$$

where $U(\zeta) \in \mathbf{C}^{m\times m}$ and $V(\zeta) \in \mathbf{C}^{n\times n}$ are unitary and

$$\Sigma(\zeta) = \mathrm{diag}(\sigma_1(\zeta), \ldots, \sigma_n(\zeta)), \quad U(\zeta) = (u_1(\zeta), \ldots, u_m(\zeta)), \quad V(\zeta) = (v_1(\zeta), \ldots, v_n(\zeta)).$$

Then for each $i \in \mathcal{S}_A(\delta_A)$, where $\mathcal{S}_A(\delta_A) = \{i : \sigma_i(\zeta) \neq 0, \zeta \in [0, \delta_A]\}$, we have that

$$\exp(-\delta_A\kappa_i^A) \leq \frac{\sigma_i(\delta_A)}{\sigma_i(0)} \leq \exp(\delta_A\kappa_i^A), \tag{2.41}$$

where

$$\kappa_i^A = \max_{\zeta\in[0,\delta_A]} \frac{|\mathrm{Re}(u_i^*(\zeta)E_Av_i(\zeta))|}{|u_i^*(\zeta)A(\zeta)v_i(\zeta)|}. \tag{2.42}$$

The above characterizations give us methods for understanding the effects of scaling upon the RSVD, QSVD, PSVD, and, of course, the ordinary SVD.

3. Effect of Structured Perturbations. In this section, we discuss the effect of common structured errors. For this part of the theory we state the results for the SVD and QSVD. Similar bounds can be derived for the PSVD and RSVD. The theory for Hermitian pencils can be written in terms of a particular QSVD problem.

3.1. Structured Perturbations of the SVD and QSVD. For the ordinary SVD, we suppose that the perturbation matrix ΔA in (2.18) has the form

$$\Delta A = \delta_A E_A$$

where

$$E_A = F_A D_A, \quad \|D_A\| = \|A\|, \quad \|F_A\| = 1.$$

Here D_A is some right scaling matrix.

Note that we do not require (2.19), that is, in general $\delta_A \neq \|\Delta A\|$. Thus the values of κ_i mean something slightly different. However, products of the form $\delta_A \kappa_i^A$ remain the same, hence the results of the analysis are the same.

Our aim is to obtain a bound of κ_i^A for all $i \in \mathcal{S}_A(\delta_A)$ as defined in Corollary 2.9. As before we let $A(\zeta), \zeta \in [0, \delta_A]$ be defined by (2.22).

We now introduce the notion of a truncated SVD. In this case, we truncate with respect to the index set $\mathcal{S}_A(\delta_A)$.

DEFINITION 3.1. *Let k be the number of indices in $\mathcal{S}_A(\delta_A)$, and let the singular values of $A(\zeta)$ whose indices are in $\mathcal{S}_A(\delta_A)$ correspond to singular values $\sigma_1(\zeta), \ldots, \sigma_k(\zeta)$. Let the truncated SVD of $A(\zeta)$ with respect to $\mathcal{S}_A(\delta_A)$ be given by*

$$A(\zeta; \delta_A) = U(\zeta)\Sigma(\zeta; \delta_A)V^*(\zeta), \quad \zeta \in [0, \delta_A],$$

where

$$\Sigma(\zeta; \delta_A) = \operatorname{diag}(\sigma_1(\zeta), \sigma_2(\zeta), \ldots, \sigma_k(\zeta), 0, \ldots, 0).$$

It is also appropriate to define the Moore-Penrose pseudoinverse of $A(\zeta; \delta_A)$. For a fixed matrix $A \in \mathbf{C}^{m\times n}$, the Moore-Penrose pseudoinverse is the unique matrix $A^\dagger \in \mathbf{C}^{n\times m}$ satisfying the four Penrose conditions

$$\begin{aligned} &1.\ AA^\dagger A = A, && 3.\ (AA^\dagger)^* = AA^\dagger, \\ &2.\ A^\dagger A A^\dagger = A^\dagger, && 4.\ (A^\dagger A)^* = A^\dagger A. \end{aligned}$$

It is easily verified that the Moore-Penrose pseudoinverse of $A(\zeta; \delta_A)$, $\zeta \in [0, \delta_A]$, is given by

$$A^\dagger(\zeta; \delta_A) = V(\zeta)\Sigma^\dagger(\zeta; \delta_A)U^*(\zeta),$$

where

$$\Sigma^\dagger(\zeta; \delta_A) = \operatorname{diag}(\sigma_1^{-1}(\zeta), \ldots, \sigma_k^{-1}(\zeta), 0, \ldots, 0).$$

We now use this form to establish global error bounds for all $\sigma_i, i \in \mathcal{S}_A(\delta_A)$.

PROPOSITION 3.2. *Let $A, \Delta A \in \mathbf{C}^{m\times n}$, and let $\Delta A = \delta_A F_A D_A$, where $\|F_A\| = 1$. Define $A(\zeta) = A + \zeta F_A D_A$ for $\zeta \in [0, \delta_A]$. Let $A(\zeta)$ have the singular value decomposition assumed in Corollary 2.9. Let $\sigma_i(\zeta), i = 1, 2, \ldots, n$ denote the singular values of $A(\zeta)$, and let $A(\zeta; \delta_A)$ be as defined in Definition 3.1. Then for each $i \in \mathcal{S}_A(\delta_A)$ (that is, for each $i = 1, 2, \ldots, k$), $\sigma_i(\delta_A)$ satisfies (2.41) with κ_i^A bounded by*

$$\kappa_i^A \le \chi_i^A = \max_{\zeta\in[0,\delta_A]} \|D_A A^\dagger(\zeta; \delta_A)u_i(\zeta)\| \le \max_{\zeta\in[0,\delta_A]} \|D_A A^\dagger(\zeta; \delta_A)\|.$$

Proof. From (2.42), for each $i \in \mathcal{S}_A(\delta_A)$ we have

$$\kappa_i^A = \max_{\zeta \in [0,\delta_A]} \frac{|\mathrm{Re}(u_i^*(\zeta) F_A D_A v_i(\zeta))|}{|u_i^*(\zeta) A(\zeta) v_i(\zeta)|}. \tag{3.1}$$

Using the fact that $\|u_i^*(\zeta) F_A\| \leq 1$ with (3.1) yields

$$\kappa_i^A \leq \max_{\zeta \in [0,\delta_A]} \frac{\|D_A v_i(\zeta)\|}{\sigma_i(\zeta)}. \tag{3.2}$$

By the definition of $A^\dagger(\zeta;\delta_A)$ we have

$$v_i(\zeta) = A^\dagger(\zeta;\delta_A) u_i(\zeta) \sigma_i(\zeta). \tag{3.3}$$

Combining (3.2) with (3.3) yields the desired result. □

The following corollary is a componentwise error bound that we might expect from singular value improvement procedures. Its proof is very similar to the scaled case.

COROLLARY 3.3. *Let $A, \Delta A \in \mathbf{C}^{m\times n}$, and let $\Delta A = \delta_A E_A$, where $|E_A| \leq |A|$. Here both the inequality and the absolute value are componentwise. Assume the rest of the hypothesis of Corollary 2.9. Then (2.41) holds for each $i \in \mathcal{S}_A(\delta_A)$ with κ_i^A bounded by*

$$\kappa_i^A \leq \rho_i^A = \max_{\zeta \in [0,\delta_A]} \| \, |A| \, |A^\dagger(\zeta;\delta_A) u_i(\zeta)| \, \| \leq \max_{\zeta \in [0,\delta_A]} \| \, |A| \, |A^\dagger(\zeta;\delta_A)| \, \|.$$

We now consider the effect of scaled and componentwise errors for the QSVD. For simplicity assume that $\mathrm{Null}(A) \cap \mathrm{Null}(B) = \{0\}$. The QSVD of (A, B) yields an expanded definition of pseudoinverse discussed in [5, 2].

DEFINITION 3.4. *The B-weighted pseudoinverse of the matrix A is unique matrix $A_B^\dagger$ that satisfies the four conditions*

$$1.\ A A_B^\dagger A = A, \qquad 2.\ A_B^\dagger A A_B^\dagger = A_B^\dagger$$
$$3.\ (A A_B^\dagger)^* = A A_B^\dagger, \quad 4.\ (B^* B A_B^\dagger A)^* = B^* B A_B^\dagger A$$

Using the QSVD, the B-weighted pseudoinverse of A and the A-weighted pseudoinverse of B are given by

$$A_B^\dagger = X \Sigma_A^\dagger U^*, \quad B_A^\dagger = X \Sigma_B^\dagger V^*.$$

Now as with the ordinary SVD, we can simply use truncated weighted pseudoinverses. We let $A(\zeta), \zeta \in [0,\delta_A]$, be given by (2.36). We then let

$$\Sigma_A(\zeta;\delta_A) = \mathrm{diag}(\alpha_1(\zeta;\delta_A), \ldots, \alpha_n(\zeta;\delta_A)),$$

where

$$\alpha_i(\zeta;\delta_A) = \begin{cases} \alpha_i(\zeta) & i \in \mathcal{S}_A(\delta_A) \\ 0 & \text{otherwise} \end{cases}.$$

We then truncate $A(\zeta)$ to obtain

$$A(\zeta;\delta_A) = U(\zeta) \Sigma_A(\zeta;\delta_A) X^{-1}(\zeta) \quad \zeta \in [0,\delta_A].$$

Thus, B-weighted pseudoinverse of $A(\zeta;\delta_A)$ is clearly given by

$$A_B^\dagger(\zeta;\delta_A) = X(\zeta)\Sigma_A^\dagger(\zeta;\delta_A)U^*(\zeta).$$

We also let $B(\xi), \xi \in [0,\delta_B]$ be given by (2.37). We then define

$$\tilde{\Sigma}_B(\xi;\delta_B) = \mathrm{diag}(\tilde{\beta}_1(\xi;\delta_B), \ldots \tilde{\beta}_n(\xi;\delta_B)),$$

where

$$\tilde{\beta}_i(\xi;\delta_B) = \begin{cases} \tilde{\beta}_i(\xi) & i \in \mathcal{S}_B(\delta_B) \\ 0 & \text{otherwise.} \end{cases}$$

Thus we truncate $B(\xi)$ to $B(\xi;\delta_B)$ giving us

$$B(\xi;\delta_B) = \tilde{V}(\xi)\tilde{\Sigma}_B(\xi;\delta_B)\tilde{X}^{-1}(\xi).$$

We then note that the $\tilde{A}$-weighted pseudoinverse of $B(\xi;\delta_B)$ is given by

$$B_{\tilde{A}}(\xi;\delta_B) = \tilde{X}(\xi)\tilde{\Sigma}_B^\dagger(\xi;\delta_B)\tilde{V}^*(\xi).$$

For the QSVD, the condition numbers κ_i^A and κ_i^B have a straightforward interpretation in terms of truncated pseudoinverses. Its proof is analogous to that for the ordinary SVD case above.

PROPOSITION 3.5. *Let (A,B) and $(\tilde{A},\tilde{B})$ be matrix pairs such that $A,\tilde{A} \in \mathbf{C}^{m\times n}$ and $B,\tilde{B} \in \mathbf{C}^{p\times n}$. Let $\Delta A = \delta_A E_A$ and $\Delta B = \delta_B E_B$. Let $E_A = F_A D_A$ and $E_B = F_B D_B$ where $\|F_A\| = \|F_B\| = 1$, and assume the rest of the hypothesis and terminology of Corollary 2.7. Define*

$$\chi_i^A = \max_{\zeta\in[0,\delta_A]} \|D_A A_B^\dagger(\zeta;\delta_A)u_i(\zeta)\|, \quad i \in \mathcal{S}_A(\delta_A),$$

$$\chi_i^B = \max_{\xi\in[0,\delta_B]} \|D_B B_{\tilde{A}}^\dagger(\xi;\delta_B)\tilde{v}_i(\xi)\|, \quad i \in \mathcal{S}_B(\delta_B).$$

Then

$$\kappa_i^A \le \chi_i^A, \quad i \in \mathcal{S}_A(\delta_A), \quad \kappa_i^B \le \chi_i^B(\delta_B), \quad i \in \mathcal{S}_B(\delta_B).$$

This is a generalization of a bound by Demmel and Veselić [3]. If D_A, D_B, A, and B have full column rank, then this becomes exactly that result. Note, however, that the character of this bound changes when either A or B has some near zero generalized singular values.

COROLLARY 3.6. *Let (A,B) and $(\tilde{A},\tilde{B})$ be matrix pairs such that $A,\tilde{A} \in \mathbf{C}^{m\times n}$ and $B,\tilde{B} \in \mathbf{C}^{p\times n}$. Let $\Delta A = \delta_A E_A$ and $\Delta B = \delta_B E_B$, where $|E_A| \le |A|$ and $|E_B| \le |B|$. Assume the rest of the hypothesis and terminology of Corollary 2.7. Let $\mathcal{S}_A(\delta_A)$ be the set of indices where $\alpha(\zeta) \neq 0, \zeta \in [0,\delta_A]$ and let $\mathcal{S}_A(\delta_A)$ be the set of indices $\tilde{\beta}_i(\xi) \neq 0, \xi \in [0,\delta_B]$. Define*

$$\rho_i^A = \max_{\zeta\in[0,\delta_A]} \|\, |A|\, |A_B^\dagger(\zeta;\delta_A)u_i(\zeta)|\, \|, \quad i \in \mathcal{S}_A(\delta_A),$$

$$\rho_i^B = \max_{\xi\in[0,\delta_B]} \|\, |B|\, |B_{\tilde{A}}^\dagger(\xi;\delta_B)\tilde{v}_i(\xi)|\, \|, \quad i \in \mathcal{S}_B(\delta_B).$$

Then

$$\kappa_i^A \le \rho_i^A, \quad i \in \mathcal{S}_A(\delta_A), \quad \textit{and} \quad \kappa_i^B \le \rho_i^B, \quad i \in \mathcal{S}_B(\delta_B).$$

Moreover,

$$\rho_i^A \leq \max_{\zeta\in[0,\delta_A]} \| \, |A| \, |A_B^\dagger(\zeta;\delta_A)| \, \|, \quad i \in \mathcal{S}_A(\delta_A),$$

$$\rho_i^B \leq \max_{\xi\in[0,\delta_B]} \| \, |B| \, |B_{\tilde{A}}^\dagger(\xi;\delta_B)| \, \|, \quad i \in \mathcal{S}_B(\delta_B).$$

The structured perturbations for the SVD and QSVD set a stage for us to give structured perturbation bounds for Hermitian generalized eigenvalue problem.

3.2. Structured Perturbations for the Hermitian Generalized Eigenvalue Problem. Veselić and Slapničar [16] describe a formalism that allow us to reduce the considering of a general Hermitian pencil to a particular QSVD. To do so, we first define the spectral absolute value of the matrix $H(\zeta)$ with respect to M. Suppose that the family of pencils $H(\zeta) - \lambda M, \zeta \in [0, \delta_H]$ has the form

$$H(\zeta) = X^{-*}(\zeta)\Lambda(\zeta)X^{-1}(\zeta), \quad M = X^{-*}(\zeta)J(\zeta)X^{-1}(\zeta), \tag{3.4}$$

where

$$X(\zeta) = (x_1(\zeta), \ldots, x_n(\zeta)),$$
$$\Lambda_H(\zeta) = \mathrm{diag}(\lambda_1(\zeta), \ldots, \lambda_n(\zeta)),$$
$$J(\zeta) = \mathrm{diag}(j_1(\zeta), \ldots, j_n(\zeta)).$$

Here

$$j_i(\zeta) = \begin{cases} 0 & \text{if } x_i(\zeta) \in \mathrm{Null}(M) \ , \\ 1 & \text{otherwise.} \end{cases}$$

As done in Veselić and Slapničar [16], we relate our problem to a positive definite eigenvalue problem.

Definition 3.7. *Let $H(\zeta), \zeta \in [0, \delta_H]$ and M be Hermitian and let M be positive semi-definite. Let the pair $(H(\zeta), M)$ have the generalized eigendecomposition in (3.4). The spectral absolute value of $H(\zeta)$ with respect to M is the matrix $H_M^\ddagger(\zeta)$ given by*

$$H_M^\ddagger = X^{-*}(\zeta)|\Lambda(\zeta)|X^{-1}(\zeta).$$

Here $|\Lambda(\zeta)| = \mathrm{diag}(|\lambda_1(\zeta)|, |\lambda_2(\zeta)|, \ldots, |\lambda_n(\zeta)|)$. If $M = I$, then we define $H^\ddagger(\zeta)$ by

$$H^\ddagger(\zeta) = H_I^\ddagger(\zeta).$$

If we let $X^{-1}(\zeta)$ have the factorization

$$X^{-1}(\zeta) = Q(\zeta)R(\zeta),$$

where $Q(\zeta)$ is unitary, then it is easily seen that

$$H_M^\ddagger(\zeta) = R^*(\zeta)(R^{-*}(\zeta)H(\zeta)R^{-1}(\zeta))^\ddagger R(\zeta).$$

This is the definition given by Veselić and Slapničar [16] for the case where M is nonsingular. We also note that for the case $M = I$, we have

$$H^\ddagger(\zeta) = \sqrt{H^2(\zeta)}$$

where $\sqrt{\cdot}$ denotes matrix square root.

We will now define a truncated version of $H_M^{\ddagger}(\zeta)$. Define $\mathcal{S}_H(\delta_H)$ as in (2.3).

DEFINITION 3.8. *We define $H_M^{\ddagger}(\zeta;\delta_H), \zeta \in [0,\delta_H]$ as the truncated spectral absolute value of $H(\zeta)$ with respect to M. It is the matrix $H_M^{\ddagger}(\zeta;\delta_H)$ such that*

$$H_M^{\ddagger}(\zeta;\delta_H) = X^{-*}|\Lambda(\zeta;\delta_H)|X^{-1},$$

where

$$|\Lambda(\zeta;\delta_H)| = \mathrm{diag}(|\lambda_1(\zeta;\delta_H)|,\ldots,|\lambda_n(\zeta;\delta_H)|)$$

and

$$\lambda_i(\zeta;\delta_H) = \begin{cases} 0, & i \in \mathcal{S}_H(\delta_H), \\ \lambda_i(\zeta), & \textit{otherwise.} \end{cases}$$

Clearly, $H_M^{\ddagger}(\zeta;\delta_H)$ is positive semi-definite. We can factor both $H_M^{\ddagger}(\zeta;\delta_H)$ and M into the form

$$H_M^{\ddagger}(\zeta;\delta_H) = C^*(\zeta;\delta_H)C(\zeta;\delta_H), \quad \zeta \in [0,\delta_H], \tag{3.5}$$

and

$$M = G^*G, \tag{3.6}$$

respectively, where

$$C(\zeta;\delta_H) = U(\zeta)\Phi(\zeta;\delta_H)X^{-1}(\zeta) \in \mathbf{C}^{m\times n}, \quad m \le n, \quad \zeta \in [0,\delta_H] \tag{3.7}$$

$$G = V(\zeta)J(\zeta)X^{-1}(\zeta) \in \mathbf{C}^{p\times n}, \quad p \le n, \quad \zeta \in [0,\delta_H].$$

In (3.7) U and V are matrices with orthonormal rows and orthonormal nontrivial columns. That is, columns of U which correspond to $i \in \mathcal{S}_H(\delta_H)$ are orthonormal, and columns of V for which $j_i(\zeta) = 1$ are orthonormal. Also, $\Phi(\zeta;\delta_H)$ satisfies

$$\Phi(\zeta;\delta_H) = \mathrm{diag}(\phi_i(\zeta;\delta_H)) = \sqrt{|\Lambda(\zeta;\delta_H)|}. \tag{3.8}$$

The form (3.7) describes the QSVD of the pair $(C(\zeta;\delta_H), G)$. This allows us to establish bounds that are similar to those in the previous section.

Clearly, the G-weighted pseudoinverse of $C(\zeta;\delta_H), \zeta \in [0,\delta_H]$, is given by

$$C_G^{\dagger}(\zeta;\delta_H) = X(\zeta)\Phi^{\dagger}(\zeta;\delta_H)U^*(\zeta), \quad \zeta \in [0,\delta_H]. \tag{3.9}$$

Using this structure, we can establish bounds on all of the eigenvalues that do not change sign under the perturbation.

THEOREM 3.9. *Let $H(\zeta), \zeta \in [0,\delta_H]$ be Hermitian and have the form (3.4). Let $(\lambda_i(\zeta), x_i(\zeta))$ be the i^{th} eigenpair of the pair $(H(\zeta), M)$ where M is Hermitian and positive semi-definite. Let $C(\zeta;\delta_H)$, $\zeta \in [0,\delta_H]$ be as defined in (3.5), let G be defined by (3.6), and let the QSVD of $(C(\zeta;\delta_H), G)$ be given by (3.7). Define $\mathcal{S}_H(\delta_H)$ as in (2.3). Then each $\lambda_i(\zeta), i \in \mathcal{S}_H(\delta_H)$, satisfies (2.4), where*

$$\kappa_i^H = \max_{\zeta\in[0,\delta_H]} \frac{|x_i^*(\zeta)E_H x_i(\zeta)|}{x_i^*(\zeta)H_M^{\ddagger}(\zeta;\delta_H)x_i(\zeta)} \tag{3.10}$$

$$= \max_{\zeta\in[0,\delta_H]} |u_i^*(\zeta)[C_G^{\dagger}(\zeta;\delta_H)]^* E_H C_G^{\dagger}(\zeta;\delta_H)u_i(\zeta)|.$$

Moreover, we have

$$\max_{i \in \mathcal{S}_H(\delta_H)} \kappa_i^H \leq \max_{\zeta \in [0,\delta_H]} \|[C_G^\dagger(\zeta;\delta_H)]^* E_H C_G^\dagger(\zeta;\delta_H)\|. \tag{3.11}$$

Proof. For each $i \in \mathcal{S}_H(\delta_H)$ considering (2.4) yields

$$\kappa_i^H = \max_{\zeta \in [0,\delta_H]} \frac{|x_i^*(\zeta) E_H x_i(\zeta)|}{x_i^*(\zeta) H_M^\ddagger(\zeta;\delta_H) x_i(\zeta)} = \max_{\zeta \in [0,\delta_H]} \frac{|x_i^*(\zeta) E_H x_i(\zeta)|}{\phi_i(\zeta)^2}.$$

Using the fact that $x_i(\zeta) = \phi_i(\zeta) C_G^\dagger(\zeta;\delta_H) u_i(\zeta)$, we have

$$\kappa_i^H \leq \max_{\zeta \in [0,\delta_H]} |u_i^*(\zeta)[C_G^\dagger(\zeta;\delta_H)]^* E_H C_G^\dagger(\zeta;\delta_H) u_i(\zeta)|,$$

which is the second equality in (3.10). Thus,

$$\kappa_i^H = \max_{\zeta \in [0,\delta_H]} \|u_i^*(\zeta)[C_G^\dagger(\zeta;\delta_H)]^* E_H C_G^\dagger(\zeta;\delta_H) u_i(\zeta)\|,$$

and classical norm inequalities yield (3.11). □

The componentwise version of Theorem 3.9 is obtained similarly as in §3.1.

The following corollary yields a bound for the case of scaled perturbations discussed by Barlow and Demmel [1]. Here E_H has the form

$$E_H = D^* F_H D, \quad \|F_H\| = 1. \tag{3.12}$$

COROLLARY 3.10. *Assume the hypothesis and terminology of Theorem 3.9. Assume that E_H has the form (3.12) and assume that $C_G(\zeta;\delta_H)$ is defined by (3.7). Then*

$$\kappa_i^H \leq \chi_i^H = \max_{\zeta \in [0,\delta_H]} \|D C_G^\dagger(\zeta;\delta_H) u_i(\zeta)\|^2, \quad i \in \mathcal{S}(\delta_H).$$

4. Error Bounds on Subspaces. We now consider the effect of structured perturbations on the eigenvectors of H. We confine our attention to the perturbed problem

$$(H + \Delta H)\tilde{x} = \tilde{\lambda}\tilde{x},$$

where ΔH has the form (3.12).

We let $H(\zeta)$ be as in (2.1), thus $\mathcal{S}(\delta_H)$ has the form

$$\mathcal{S}(\delta_H) = \{i : \lambda_i(\zeta) \neq 0, \zeta \in [0,\delta_H]\} \tag{4.1}$$

and its set complement is

$$\mathcal{S}^c(\delta_H) = \{i : \lambda_i(\zeta) = 0, \text{ for some } \zeta \in [0,\delta_H]\}. \tag{4.2}$$

Suppose that $\mathcal{S}(\delta_H)$ has k elements and that $\mathcal{S}^c(\delta_H)$has $n-k$ elements. Let $X_1, \tilde{X}_1 \in \mathbf{C}^{n\times k}$ be the matrices of eigenvectors of H and $H + \Delta H$ associated with $\mathcal{S}(\delta_H)$ and let $X_2, \tilde{X}_2 \in \mathbf{C}^{n\times(n-k)}$ be the matrices of eigenvectors associated with $\mathcal{S}^c(\delta_H)$.

We now define several forms of *relative gaps*:

$$relgap(\lambda,\mu) = \frac{\lambda - \mu|}{|\lambda\mu|^{1/2}}, \quad \mu,\lambda \neq 0, \tag{4.3}$$

$$relgap(\lambda_i;\delta_H) = \min_{j\neq i}\min_{j\in\mathcal{S}_H(\delta_H)} relgap(\lambda_i,\lambda_j(\delta_H)), \tag{4.4}$$

$$relgap(\Gamma,\Lambda) = \min_{i,j} relgap(\lambda_i,\gamma_j), \quad \Lambda = \mathrm{diag}(\lambda_i), \quad \Gamma = \mathrm{diag}(\gamma_j),$$

$$relgap_0(\tilde{\lambda}_i;\delta_H) = \|(\tilde{\lambda}_i I - \Lambda_2)^{-1}|\lambda_i|\,\|,$$

$$relgap_0(\Gamma;\delta_H) = \min_j relgap_0(\gamma_j;\delta_H).$$

The first definition (4.3) is just that from Barlow and Demmel [1]. The second definition (4.4) is the distance of a particular non–zero eigenvalue from all of the other non–zero eigenvalues. The third definition generalized the first to sets of eigenvalues. The fourth and fifth definitions generalize (4.4) to distance from the set of near zero eigenvalues.

Let $H^{\ddagger}(\zeta)$ and $C(\zeta)$ be defined by Definition 3.7 and (3.7), respectively. Note that, since the eigenvector matrix $X(\zeta)$ is unitary, we can also set $U(\zeta) = X^{-*}(\zeta) = X(\zeta)$ in (3.7). We will also need the values

$$\begin{aligned}
\chi_i^H &= \max_{\zeta\in[0,\delta_H]} \|DC_G^{\dagger}(\zeta;\delta_H)u_i(\zeta)\|,\\
\chi^H &= \max_{\zeta\in[0,\delta_H]} \|DC_G^{\dagger}(\zeta;\delta_H)\|,\\
\chi_0 &= \frac{\|DX_2\|^2}{\min_{i\in\mathcal{S}_H(\delta_H)}|\lambda_i(\delta_H)|},\\
\chi_F^H &= \max_{\zeta\in[0,\delta_H]} \|DC_G^{\dagger}(\zeta;\delta_H)\|_F.
\end{aligned} \tag{4.5}$$

We can write down the following theorem on the error in the i^{th} eigenvector of H.

THEOREM 4.1. *Let $H,\delta H \in \mathbf{C}^{n\times n}$ be Hermitian and let $\mathcal{S}(\delta_H)$ and $\mathcal{S}^c(\delta_H)$ be defined by (4.1) and (4.2), respectively. For $i \in \mathcal{S}(\delta_H)$, let $x_i,\tilde{x}_i$ be the i^{th} eigenvectors of H and $H+\Delta H$, respectively. Let X_{i1} be the matrix X_1 with x_i excluded. If ΔH has the form (3.12) then*

$$\|X_{i1}^*\tilde{x}_i\| \leq \frac{\delta_H}{relgap(\lambda_i;\delta_H)}\sqrt{\chi^H\chi_i^H},$$

$$\|X_2^*\tilde{x}_i\| \leq \frac{\delta_H}{relgap_0(\tilde{\lambda}_i;\delta_H)}\sqrt{\chi_0\chi_i^H},$$

$$\|X_2^*\tilde{X}_1\|_F \leq \frac{\delta_H}{relgap_0(\tilde{\Lambda}_1;\delta_H)}\sqrt{\chi_0\chi_F^H},$$

where χ_i^H,χ^H,χ_0 and χ_F^H are defined by (4.5).

Proof. We have that

$$X_{i1}^*(H+\Delta H)\tilde{x}_i = \tilde{\lambda}_i X_{i1}^*\tilde{x}_i$$

and, thus,

$$\begin{aligned}
X_{i1}^*\tilde{x}_i &= (\tilde{\lambda}_i I - \Lambda_{i1})^{-1}X_{i1}^*\Delta H\tilde{x}_i\\
&= (\tilde{\lambda}_i I - \Lambda_{i1})^{-1}X_{i1}^*D^*F_H D\tilde{x}_i.
\end{aligned}$$

Since

$$X_{i1} = C^{\dagger}(0;\delta_H)U_{i1}|\Lambda_{i1}|^{1/2}, \quad \tilde{\lambda}_i = C^{\dagger}(\delta_H;\delta_H)\tilde{u}_i|\lambda_i|^{1/2},$$

we have

$$\|X_{i1}^* \tilde{x}_i\| \le \|(\tilde{\lambda}_i I - \Lambda_{i1})^{-1}(|\Lambda_{i1}|\,|\tilde{\lambda}_i|)^{1/2}\|\,\|DC^\dagger(0;\delta_H)\|\,\|DC^\dagger(\delta_H;\delta_H)\tilde{u}_i\|$$

$$= \frac{\delta_H\left(\chi^H \chi_i^H\right)^{1/2}}{relgap(\lambda_i;\delta_H)}.$$

Likewise,

$$X_2^* \tilde{x}_i = (\tilde{\lambda}_i I - \Lambda_2)^{-1} X_2^* \Delta H \tilde{x}_i.$$

Thus,

$$\|X_2^* \tilde{x}_i\| \le \delta_H \|(\tilde{\lambda}_i I - \Lambda_2)^{-1}|\tilde{\lambda}_i|\,\|\left(\chi_0 \chi_i^H\right)^{1/2} = \frac{\delta_H\left(\chi_0 \chi_i^H\right)^{1/2}}{relgap_0(\tilde{\lambda}_i;\delta_H)}.$$

A straightforward computation yields

$$\|X_2^* \tilde{X}_1\|_F \le \frac{\delta_H\left(\chi_0 \chi_F^H\right)^{1/2}}{relgap_0(\tilde{\Lambda}_1;\delta_H)},$$

as desired. □

This can be generalized to (1.3) with M positive definite if we substitute an M–weighted norm for the Euclidean. Natural analogs of the results in this section are easily stated.

A result that is useful for clusters of eigenvalues is given next.

THEOREM 4.2. *Assume the hypothesis and terminology of Theorem 4.1. Let* Λ_1 *and* $\tilde{\Lambda}_1$ *be partitioned as*

$$\Lambda_1 = \operatorname{diag}(\Lambda_{11}, \Lambda_{12}), \quad \tilde{\Lambda}_1 = \operatorname{diag}(\tilde{\Lambda}_{11}, \tilde{\Lambda}_{12}),$$

and let X_1 *and* $\tilde{X}_1$ *be partitioned conformally as*

$$X_1 = \begin{array}{c} \begin{array}{cc} p & k-p \end{array} \\ \begin{pmatrix} X_{11} & X_{12} \end{pmatrix} \end{array}, \quad \tilde{X}_1 = \begin{array}{c} \begin{array}{cc} p & k-p \end{array} \\ \begin{pmatrix} \tilde{X}_{11} & \tilde{X}_{12} \end{pmatrix} \end{array}.$$

Then

$$\|X_{11}^* \tilde{X}_{12}\|_F \le \frac{\delta_H \chi_F^H}{relgap(\Lambda_{11}, \tilde{\Lambda}_{12})}. \tag{4.6}$$

Proof. Let x_i be a column of X_{11} and let $\tilde{x}_j$ be a column of $\tilde{X}_{12}$. Then a straightforward computation yields

$$|x_i^* \tilde{x}_j| \le \frac{|x_i^* \Delta H \tilde{x}_j|}{|\tilde{\lambda}_j - \lambda_i|} \le \delta_H \frac{|x_i^* D^* F_H D \tilde{x}_j|}{|\tilde{\lambda}_j - \lambda_i|}$$

$$\le \delta_H \frac{\|Dx_i\|\,\|D\tilde{x}_j\|}{|\tilde{\lambda}_j - \lambda_i|} \le \delta_H \frac{\sqrt{\chi_i^H \chi_j^H}}{relgap(\lambda_i, \tilde{\lambda}_j)}.$$

Summing over i and j yields (4.6). □

An important type of perturbation for singular value problems for positive definite H is when H is perturbed through its factors, that is

$$H = G^* G, \quad H + \Delta H = (G + \Delta G)^*(G + \Delta G). \tag{4.7}$$

THEOREM 4.3. *Let H and $H+\Delta H$ have the form (4.7). Assume that for some $\eta<1$, G and ΔG satisfy*

$$\|\Delta G x\| \leq \eta \|Gx\|, \quad x \in \mathbf{C}^n.$$

If (λ_i, x_i) and $(\tilde{\lambda}_i, \tilde{x}_i)$ are the i^{th} eigenpairs of H and $H+\Delta H$, respectively, then

$$|x_j^* \tilde{x}_i| \leq \frac{\eta[1+(1-\eta)^{-1}]}{relgap(\lambda_i, \tilde{\lambda}_j)}. \tag{4.8}$$

Proof. We have that

$$x_j^*(G+\Delta G)^*(G+\Delta G)\tilde{x}_i = \tilde{\lambda}_i x_j^* \tilde{x}_i.$$

That may be rewritten as

$$x_j^* G^* \Delta G \tilde{x}_i + x_j^* \Delta G (G+\Delta G)\tilde{x}_j = (\tilde{\lambda}_i - \lambda_j) x_j^* \tilde{x}_i,$$

and again as

$$|\lambda_j|^{1/2} u_j^* \Delta G \tilde{x}_j + |\tilde{\lambda}_i|^{1/2} x_j^* \Delta G^* \tilde{u}_i = (\tilde{\lambda}_i - \lambda_j) x_j^* \tilde{x}_i.$$

We note that

$$\|\Delta G x_i\| \leq \eta \|G x_i\| = \eta |\lambda_i|^{1/2},$$

and

$$\|\Delta G \tilde{x}_j\| \leq \eta \|G \tilde{x}_j\| = \eta [\, \|(G+\Delta G)\tilde{x}_j\| + \|\Delta G \tilde{x}_j\|\,].$$

Thus,

$$\|\Delta G \tilde{x}_j\| \leq \eta (1-\eta)^{-1} |\tilde{\lambda}_j|^{1/2}.$$

An application of the Cauchy–Schwarz inequality and some algebra yields (4.8). □

5. Examples.

In this section we illustrate our results on several examples.

We give examples for structured perturbations of §3, in particular for the relative componentwise perturbations of the type

$$\Delta A = \delta_A E_A, \quad |E_A| \leq |A|.$$

Such perturbations are highly interesting since they appear during various numerical algorithms for eigenvalue and singular value problems [1, 3, 7, 15, 16, 17]. Such perturbations are sometimes called *floating-point* perturbations [16].

The first two examples deal with the singular value decomposition and illustrate Corollary 3.3.

EXAMPLE 5.1. Let A be a product of a well-conditioned matrix and a strong column scaling,

$$A = \begin{pmatrix} -2\cdot 10^{40} & 7\cdot 10^{20} & 7 \\ -8\cdot 10^{40} & -7\cdot 10^{20} & -6 \\ -7\cdot 10^{40} & 2\cdot 10^{20} & 2 \end{pmatrix} = \begin{pmatrix} -2 & 7 & 7 \\ -8 & -7 & -6 \\ -7 & 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 10^{40} & & \\ & 10^{20} & \\ & & 1 \end{pmatrix} = BD.$$

Let $\Delta A = \delta_A E_A$ where $\delta_A = 10^{-6}$ and

$$E_A = \begin{pmatrix} 7\cdot 10^{39} & -1\cdot 10^{20} & 3 \\ -3\cdot 10^{40} & 1\cdot 10^{20} & -1 \\ -9\cdot 10^{39} & 3\cdot 10^{19} & 0.4 \end{pmatrix}.$$

Let $\delta\sigma_i = \sigma_i(A + \Delta A) - \sigma_i(A)$. The singular values of A are (properly rounded)

$$(\sigma_1, \sigma_2, \sigma_3) = (1.08 \cdot 10^{41}, 9.76 \cdot 10^{20}, 0.426),$$

and the relative changes in the singular values are

$$\left(\frac{|\delta\sigma_1|}{\sigma_1}, \frac{|\delta\sigma_2|}{\sigma_2}, \frac{|\delta\sigma_3|}{\sigma_3}\right) = (2.4 \cdot 10^{-7}, 1.6 \cdot 10^{-7}, 3.5 \cdot 10^{-6}).$$

Both singular value decompositions, of A and $A+\Delta A$, are computed by the one-sided Jacobi method whose sufficiently high accuracy is guaranteed by the analysis of Demmel and Veselić [3].

Since $|E_A| \leq |A|$, we can apply Corollary 3.3. We compute the first order approximations of the corresponding bounds, that is,

$$\frac{|\delta\sigma_i|}{\sigma_i} \leq \delta_A \| \, |A| \, |A^\dagger u_i| \, \| \leq \delta_A \| \, |A| \, |A^\dagger| \, \|. \tag{5.1}$$

Note that, since the scaling D factors out, we can use B instead of A in the above formulae, which makes the computation of the inverse much more accurate. The bounds obtained by the first inequality in (5.1) are

$$\left(\frac{|\delta\sigma_1|}{\sigma_1}, \frac{|\delta\sigma_2|}{\sigma_2}, \frac{|\delta\sigma_3|}{\sigma_3}\right) \leq (1 \cdot 10^{-6}, 1.25 \cdot 10^{-6}, 4.6 \cdot 10^{-5}).$$

The second inequality in (5.1) gives

$$\max_{i=1,2,3} \frac{|\delta\sigma_i|}{\sigma_i} \leq 4.6 \cdot 10^{-5}.$$

The similar bound is obtained by the perturbation theory by Demmel and Veselić [3],

$$\max_{i=1,2,3} \frac{|\delta\sigma_i|}{\sigma_i} \leq n\delta_A \| \, [A \operatorname{diag}^{-1}(\|A_{:i}\|)]^{-1} \| \leq 9.5 \cdot 10^{-5}. \tag{5.2}$$

EXAMPLE 5.2. Let δ_A and E_A be as in Example 5.1, and let

$$A = \begin{pmatrix} -2 \cdot 10^{40} & 7 \cdot 10^{20} & 7 \\ -8 \cdot 10^{40} & -6.0001 \cdot 10^{20} & -6 \\ -7 \cdot 10^{40} & 2 \cdot 10^{20} & 2 \end{pmatrix}.$$

This matrix differs from the one in Example 5.1 only in the element A_{22}, which is chosen to make the last two column vectors of A nearly parallel. The singular values of A are

$$(\sigma_1, \sigma_2, \sigma_3) = (1.08 \cdot 10^{41}, 9.25 \cdot 10^{20}, 0.45),$$

and the relative changes in the singular values are

$$\left(\frac{|\delta\sigma_1|}{\sigma_1}, \frac{|\delta\sigma_2|}{\sigma_2}, \frac{|\delta\sigma_3|}{\sigma_3}\right) = (2.5 \cdot 10^{-7}, 1.6 \cdot 10^{-7}, 3.0 \cdot 10^{-2}).$$

The bounds obtained by the first inequality in (5.1) are

$$\left(\frac{|\delta\sigma_1|}{\sigma_1}, \frac{|\delta\sigma_2|}{\sigma_2}, \frac{|\delta\sigma_3|}{\sigma_3}\right) \leq (1 \cdot 10^{-6}, 1.2 \cdot 10^{-6}, 4.2 \cdot 10^{-1}).$$

This shows that our bounds are local and nearly optimal. The second inequality in (5.1) and the Demmel-Veselić bound (5.2),

$$\max_{i=1,2,3} \frac{|\delta\sigma_i|}{\sigma_i} \leq 4.2 \cdot 10^{-1} \quad \text{and} \quad \max_{i=1,2,3} \frac{|\delta\sigma_i|}{\sigma_i} \leq 8.9 \cdot 10^{-1},$$

respectively, both cover only the worst case.

The following two examples illustrate the application of Theorem 3.9 to componentwise perturbed Hermitian eigenvalue problem. Both examples were also analyzed in [16].

EXAMPLE 5.3. Let

$$H = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 10^{-8} \end{pmatrix}.$$

Let $\Delta H = \delta_H E_H$, where $\delta_H = 10^{-5}$ and

$$E_H = \begin{pmatrix} 0.3 & -0.5 & 0.4 \\ -0.5 & 0 & 0 \\ 0.4 & 0 & -6 \cdot 10^{-12} \end{pmatrix}.$$

Thus, $|E_H| \leq |H|$. Let $\delta\lambda_i = \lambda_i(H + \Delta H) - \lambda_i(H)$. The eigenvalues of H are (properly rounded)

$$(\lambda_1, \lambda_2, \lambda_3) = (2, -1, 5 \cdot 10^{-9}),$$

and the relative changes in the eigenvalues are

$$\left(\frac{|\delta\lambda_1|}{\lambda_1}, \frac{|\delta\lambda_2|}{\lambda_2}, \frac{|\delta\lambda_3|}{\lambda_3} \right) = (6.7 \cdot 10^{-7}, 1.7 \cdot 10^{-6}, 9.0 \cdot 10^{-6}).$$

We want to apply Theorem 3.9 with $M = I$. Since the eigenvector matrix $X(\zeta)$ is itself unitary, we can take $U(\zeta) = V(\zeta) = X^{-*}(\zeta) = X(\zeta)$ in (3.7), which implies $C(\zeta; \delta_H) = (H^{\ddagger})^{1/2}, G(\zeta) = I$. The first order approximations of the bounds from Theorem 3.9 are

$$\frac{|\delta\lambda_i|}{\lambda_i} \leq \delta_H |x_i^* (H^{\ddagger})^{-1/2}| \, |H| \, |(H^{\ddagger})^{-1/2} x_i| \leq \delta_H \| \, |(H^{\ddagger})^{-1/2}| \, |H| \, |(H^{\ddagger})^{-1/2}| \, \|. \tag{5.3}$$

The bounds obtained by the first inequality in (5.3) are

$$\left(\frac{|\delta\lambda_1|}{\lambda_1}, \frac{|\delta\lambda_2|}{\lambda_2}, \frac{|\delta\lambda_3|}{\lambda_3} \right) \leq (1 \cdot 10^{-6}, 1.7 \cdot 10^{-5}, 3.0 \cdot 10^{-5}).$$

On the other hand, the bound obtained from the second inequality in (5.3),

$$\max_{i=1,2,3} \frac{|\delta\lambda_i|}{\lambda_i} \leq 1.9 \cdot 10^4,$$

and the bound from Veselić and Slapničar [16],

$$\max_{i=1,2,3} \frac{|\delta\lambda_i|}{\lambda_i} \leq n\delta_H \| (D^{-1} H^{\ddagger} D^{-1})^{-1} \| \leq 6.0 \cdot 10^3, \tag{5.4}$$

where $D = \operatorname{diag}(\sqrt{H_{ii}^{\ddagger}})$, are both useless.

EXAMPLE 5.4. Another interesting example is the following: let $H = DAD$, where

$$A = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 10^8 & & & \\ & 1 & & \\ & & 1 & \\ & & & 10^8 \end{pmatrix}.$$

The eigenvector matrix of H is

$$X = \begin{pmatrix} 1/\sqrt{2} & 1/2 & 1/2 & 0 \\ 0 & -1/2 & 1/2 & 1/\sqrt{2} \\ 0 & -1/2 & 1/2 & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/2 & 1/2 & 0 \end{pmatrix}.$$

Let $\Delta H = \delta_H E_H$, where $\delta_H = 10^{-6}$ and $E = 0.5 \cdot 10^{16} ww^T$, $w = (\, 1 \quad 0 \quad 0 \quad 1\,)^T$. Thus, $|E_H| \leq |H|$. The eigenvalues of H are

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (2 \cdot 10^{16}, 2 \cdot 10^8, -2 \cdot 10^8, 2),$$

and the relative changes in the eigenvalues are

$$\left(\frac{|\delta\lambda_1|}{\lambda_1}, \frac{|\delta\lambda_2|}{\lambda_2}, \frac{|\delta\lambda_3|}{\lambda_3}, \frac{|\delta\lambda_4|}{\lambda_4} \right) = (4 \cdot 10^{-16}, 49, 0.98, 1.1 \cdot 10^{-10}).$$

We see that the middle eigenvalues are very sensitive. The bounds obtained by the first inequality in (5.3) are

$$\left(\frac{|\delta\lambda_1|}{\lambda_1}, \frac{|\delta\lambda_2|}{\lambda_2}, \frac{|\delta\lambda_3|}{\lambda_3}, \frac{|\delta\lambda_4|}{\lambda_4} \right) \leq (10^{-6}, 50, 50, 10^{-6}),$$

and clearly show the different sensitivity of outer and inner eigenvalues. The bounds obtained from the second inequality in (5.3) and (5.4),

$$\max_{i=1,2,3,4} \frac{|\delta\lambda_i|}{\lambda_i} \leq 100, \quad \text{and} \quad \max_{i=1,2,3,4} \frac{|\delta\lambda_i|}{\lambda_i} \leq 200,$$

respectively, are useless.

The last example illustrates Theorem 3.9 on a matrix pair (H, M).

EXAMPLE 5.5. Let $H = D_H A^T \Sigma A D_H$ and $M = D_M B^T B D_M$, where

$$D_H = \operatorname{diag}(\, 10^8 \quad 10^4 \quad 10 \quad 10 \quad 1\,), \quad \Sigma = \operatorname{diag}(\, -1 \quad -1 \quad 1 \quad 1\,),$$
$$D_M = \operatorname{diag}(\, 10^{-4} \quad 10^{-2} \quad 10^{-2} \quad 10^{-1} \quad 1\,),$$

and

$$A = \begin{pmatrix} -3 & -5 & -5 & 0 & 2 \\ 4 & 2 & -2 & -4 & -5 \\ -1 & -1 & 1 & 1 & 1 \\ 1 & 5 & 3 & 1 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & -2 & -4 & -4 & -2 \\ -3 & 4 & 4 & 4 & 0 \\ -1 & 1 & 2 & 0 & -2 \end{pmatrix}.$$

Thus, H is indefinite singular of rank four, M is semi-definite of rank three, and H and M are scaled in opposite directions. Altogether,

$$H = \begin{pmatrix} -2.3 \cdot 10^{17} & -1.7 \cdot 10^{13} & -5.0 \cdot 10^9 & 1.6 \cdot 10^{10} & 2.8 \cdot 10^9 \\ -1.7 \cdot 10^{13} & -3.0 \cdot 10^8 & -7.0 \cdot 10^5 & 1.2 \cdot 10^6 & 3.4 \cdot 10^5 \\ -5.0 \cdot 10^9 & -7.0 \cdot 10^5 & -1.9 \cdot 10^3 & -4.0 \cdot 10^2 & 1.0 \cdot 10^2 \\ 1.6 \cdot 10^{10} & 1.2 \cdot 10^6 & -4.0 \cdot 10^2 & -1.4 \cdot 10^3 & -1.6 \cdot 10^2 \\ 2.8 \cdot 10^9 & 3.4 \cdot 10^5 & 1.0 \cdot 10^2 & -1.6 \cdot 10^2 & -1.9 \cdot 10^1 \end{pmatrix},$$

$$M = \begin{pmatrix} 1.9\cdot 10^{-7} & -1.9\cdot 10^{-5} & -2.6\cdot 10^{-5} & -2.4\cdot 10^{-4} & -4.0\cdot 10^{-4} \\ -1.9\cdot 10^{-5} & 2.1\cdot 10^{-3} & 2.6\cdot 10^{-3} & 2.4\cdot 10^{-2} & 2.0\cdot 10^{-2} \\ -2.6\cdot 10^{-5} & 2.6\cdot 10^{-3} & 3.6\cdot 10^{-3} & 3.2\cdot 10^{-2} & 4.0\cdot 10^{-2} \\ -2.4\cdot 10^{-4} & 2.4\cdot 10^{-2} & 3.2\cdot 10^{-2} & 3.2\cdot 10^{-1} & 8.0\cdot 10^{-1} \\ -4.0\cdot 10^{-4} & 2.0\cdot 10^{-2} & 4.0\cdot 10^{-2} & 8.0\cdot 10^{-1} & 8.0\cdot 10^{0} \end{pmatrix}.$$

The eigenvector matrix of the pair (H, M) is (properly rounded)

$$X = \begin{pmatrix} 1.000 & -7.396\cdot 10^{-5} & -4.802\cdot 10^{-6} & 1.857\cdot 10^{-7} & 2.183\cdot 10^{-8} \\ -2.528\cdot 10^{-3} & 1.000 & 3.010\cdot 10^{-2} & -2.964\cdot 10^{-4} & -6.718\cdot 10^{-5} \\ 3.736\cdot 10^{-3} & 5.002\cdot 10^{-1} & 8.661\cdot 10^{1} & -8.412\cdot 10^{-1} & -5.038\cdot 10^{-2} \\ 6.292\cdot 10^{-4} & -1.501\cdot 10^{-1} & -1.009\cdot 10^{1} & 2.133 & 1.680\cdot 10^{-1} \\ -2.528\cdot 10^{-5} & 1.000\cdot 10^{-2} & 5.756\cdot 10^{-1} & -2.372\cdot 10^{-1} & 3.359\cdot 10^{-1} \end{pmatrix}.$$

We have

$$\begin{aligned} X^*HX &= \operatorname{diag}(\,-2.3\cdot 10^{17} \quad 9.57\cdot 10^{8} \quad -1.3019\cdot 10^{7} \quad 7.7388 \quad 2.6\cdot 10^{-15}\,), \\ X^*MX &= \operatorname{diag}(\,-3.6\cdot 10^{-23} \quad 1.1\cdot 10^{-18} \quad 1 \quad 1 \quad 1\,). \end{aligned}$$

We conclude that $\mathcal{S}_H(\delta_H) = \{3, 4\}$,

$$\Lambda(0;\delta_H) = \operatorname{diag}(\,0 \quad 0 \quad -1.3019\cdot 10^{7} \quad 7.7388 \quad 0\,), \quad J(\zeta) = \operatorname{diag}(\,0 \quad 0 \quad 1 \quad 1 \quad 1\,),$$

where $\Lambda(0;\delta_H)$ and $J(\zeta)$ are defined by Definition 3.8 and (3.4), respectively.

Let us perturb H to $H + \Delta H$ with $\Delta H = \delta_H E_H$, where $\delta_H = 10^{-6}$ and

$$E_H = \begin{pmatrix} -1\cdot 10^{17} & 3\cdot 10^{12} & 9\cdot 10^{8} & 7\cdot 10^{9} & -3\cdot 10^{8} \\ 3\cdot 10^{12} & -4\cdot 10^{6} & 2\cdot 10^{5} & -3\cdot 10^{5} & 1\cdot 10^{5} \\ 9\cdot 10^{8} & 2\cdot 10^{5} & 9\cdot 10^{2} & 8\cdot 10^{1} & -4\cdot 10^{1} \\ 7\cdot 10^{9} & -3\cdot 10^{5} & 8\cdot 10^{1} & 4\cdot 10^{2} & 2\cdot 10^{1} \\ -3\cdot 10^{8} & 1\cdot 10^{5} & -5\cdot 10^{1} & 2\cdot 10^{1} & -6 \end{pmatrix}.$$

Thus, $|E_H| \le |H|$. The relative changes in the eigenvalues λ_i, $i \in \mathcal{S}_H(\delta_H)$, are

$$\left(\frac{|\delta\lambda_3|}{\lambda_3}, \frac{|\delta\lambda_4|}{\lambda_4}\right) = (3.5\cdot 10^{-7}, 5.3\cdot 10^{-4}).$$

The first order approximations of the bounds from Theorem 3.9 are

$$\begin{aligned} \frac{|\delta\lambda_i|}{\lambda_i} &\le \delta_H |u_i^*(0)[C_G^\dagger(0;\delta_H)]^*|\,|H|\,|C_G^\dagger(0;\delta_H)u_i(0)| \\ &\le \delta_H \|\,|[C_G^\dagger(0;\delta_H)]^*|\,|H|\,|C_G^\dagger(0;\delta_H)|\,\|, \end{aligned} \tag{5.5}$$

where $U(0)$ and $C_G^\dagger(0;\delta_H)$ are defined by (3.9), (3.8) and (3.7). We can take

$$U = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

in which case

$$[C_G^\dagger(0;\delta_H)]^* = \begin{pmatrix} -1.331\cdot 10^{-9} & 8.341\cdot 10^{-6} & 2.400\cdot 10^{-2} & -2.796\cdot 10^{-3} & 1.595\cdot 10^{-4} \\ 6.675\cdot 10^{-8} & -1.065\cdot 10^{-4} & -3.024\cdot 10^{-1} & 7.667\cdot 10^{-1} & -8.525\cdot 10^{-2} \end{pmatrix}.$$

The bounds obtained from the first inequality in (5.5) are

$$\left(\frac{|\delta\lambda_3|}{\lambda_3}, \frac{|\delta\lambda_4|}{\lambda_4}\right) \le (2.7\cdot 10^{-6}, 4.6\cdot 10^{-3}),$$

and the bound obtained from the second inequality in (5.5) is

$$\max_{i=3,4} \frac{|\delta\lambda_i|}{\lambda_i} \leq 4.6 \cdot 10^{-3}.$$

Note that choosing $C_G^{\dagger}(0;\delta_H)$ with another $U(0)$ in (3.9) would yield the same bounds.

6. Conclusion. The above examples lead us to some observations about the perturbation bounds in this paper.

Firstly, we can often obtain meaningful relative error bounds on eigenvalues of numerically singular matrices as long as those eigenvalues are bounded away from zero. If these "non–zero" eigenvalues are well–behaved, the subspace associated with the "zero" eigenvalues is also well–behaved.

Secondly, we note that much of the progress in structured perturbation theory has had to do with the SVD and its generalizations. We note that many of our structured perturbation results on the Hermitian eigenvalue problem are characterized in terms of the generalized SVD. In many circumstances, this is a more appropriate approach.

Finally, the error bounds on individual eigenvalues and vectors tend to be tighter, sometimes much tighter, than the global error bounds for all of the eigenvalues of the matrix given in [1] or [16]. Moreover, they are easier to generalize to large classes of eigenvalue problems.

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