Relative Perturbation Theory for Hyperbolic Singular Value Problem*

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Abstract

We give relative perturbation bounds for singular values and perturbation bounds for singular subspaces of a hyperbolic singular value problem for the pair (G, J), where G is a full rank matrix and J is a diagonal matrix of signs. We consider two types of relative perturbations: $G + \delta G = (B + \delta B)D$ and $G + \delta G = \bar{D}(\bar{B} + \delta \bar{B})$, depending whether G has full column or full row rank, respectively. In both cases we also consider relative element-wise perturbations of G which typically occur in numerical computations.

AMS classification: 15A18; 15A22; 15A42
Keywords: Hyperbolic singular value problems; Perturbation theory

1 Introduction

Let G be $m \times n$ full rank matrix, and let J be $n \times n$ diagonal matrix of signs, that is $J_{ii} \in \{-1, 1\}$. The hyperbolic singular value decomposition (HSVD) of the pair (G, J) is given by [10, 20]

$$G = U \Sigma V^{-1}.$$

Here U is unitary, Σ is non-negative diagonal $m \times n$ matrix, and

$$V^* J V = \bar{J},$$

where $\bar{J}$ is some permutation of J. Diagonal entries of Σ are the hyperbolic singular values of the pair (G, J), and the columns of U and V are the left (unitary) and right

*This work was supported by the grant 037012 from the Croatian Ministry of Science and Technology.

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(hyperbolic) singular vectors, respectively. The matrix V is also called the hyperexchange matrix, and its columns are called the hyperbolic singular vectors. Also, if G is has full column rank, then $V^*JV = J$, in which case we say that V is J-unitary or hypernormal with respect to J [10].

Clearly, for $J = I$ the HSVD is equal to the classical singular value decomposition since V is unitary. There are several similarities between the HSVD and the classical SVD. First, the HSVD can be written in economical form depending upon dimension. Second, the HSVD is closely related to two eigenvalue problems: the Hermitian eigenvalue problem

$$GJG^* = U\Lambda U^*, \qquad \Lambda = \Sigma J \Sigma^*, \tag{1}$$

and the J-Hermitian or hyperbolic eigenvalue problem for the pair (G^*G, J) [12],

$$V^*G^*GV = \Sigma^*\Sigma, \qquad V^*JV = \bar{J}. \tag{2}$$

Notice that for general G, Σ is not always diagonal. The necessary and sufficient condition for a diagonal Σ according to [20, Remark 5] is

$$\operatorname{rank} GJG^* = \operatorname{rank} G,$$

that is, the matrix GJG^* is required to be be non-singular. A similar assumption is used in [18] and is used here, as well. Also, the hyperbolic eigenvalue problem (2) can be viewed as the eigenvalue problem for the non-Hermitian matrix (see [19, 18])

$$\hat{H} = JG^*G.$$

In [10] the HSVD was proposed as a more suitable way for numerical solution of the downdating problem: the Hermitian eigenvalue problem for the matrix

$$AA^* - BB^*, \qquad A \in \mathbf{C}^{m\times k}, \quad B \in \mathbf{C}^{m\times l},$$

can be solved as the hyperbolic singular value problem for the pair (G, J), where

$$G = \begin{bmatrix} A & B \end{bmatrix}, \qquad J = \begin{bmatrix} I_k & \\ & -I_l \end{bmatrix}.$$

The HSVD is also part of the highly accurate algorithm for symmetric eigenvalue problem [11]. In this algorithm the given matrix H is first factorized as $H = GJG^T$, which is followed by the computation of the HSVD of the pair (G, J). Similar application of the HSVD is found in the eigenvalue decomposition algorithm for definite matrix pairs [17].

In this paper we derive the bounds for hyperbolic singular values $\sigma_i = \Sigma_{ii}$ and for the left and right singular vectors. For hyperbolic singular values we present relative perturbation bounds of Weyl and Wielandt-Hoffman type, and for singular vectors we give bounds in terms of $\sin\Theta$ theorems for the subspaces spanned by columns of U and V, respectively. The cases when G has full column and full row rank are treated separately. Clearly, a square, non-singular G is included in both cases. In the full column rank case we consider the right scaling $G = BD$, thus assuming the perturbation of the form $G + \delta G = (B + \delta B)D$, and in the full row rank case we consider the left scaling $G = \bar{D}\bar{B}$, thus assuming the perturbation of the form $G + \delta G = \bar{D}(\bar{B} + \delta\bar{B})$. In both cases our bounds can be applied to the important case of relative element-wise perturbations of the form $|\delta G_{ij}| \leq \varepsilon |G_{ij}|$, which typically occurs in numerical computations. Since the HSVD is closely related to the eigenvalue problems (1) and (2), some of our bounds follow from the existing results from [12, 16, 18, 19].

Throughout the paper all the perturbed quantities will be denoted by tilde. For example, the hyperbolic singular values of the pair (G, J) will be denoted by σ_i, and the hyperbolic singular values of the perturbed pair $(\widetilde{G}, J)$ will be denoted by $\widetilde{\sigma}_i$. In Sections 2.1 and 2.2, which deal with the perturbation bounds for hyperbolic singular values, we assume that σ_i and $\widetilde{\sigma}_i$ are in the increasing order. In Sections 2.2 and 3.2, which deal with the perturbation bounds for hyperbolic singular vectors, σ_i and $\widetilde{\sigma}_i$ can be in any order, but the ordering must be the same. $A^\dagger$ denotes the pseudoinverse of A, $\|\cdot\|$ denotes the spectral matrix norm, $\|\cdot\|_F$ denotes the Frobenius norm, and $\kappa(A) = \|A\|\,\|A^\dagger\|$ denotes the spectral condition number. Similarly to the Matlab notation, $A_{:,k}$ denotes the k-th column of A, $A_{k,:}$ denotes the k-th row of A, and $A_{:,l:k}$ denotes the columns l to k of A.

The rest of the paper is organized as follows. In Section 2 we consider G with full column rank. We first give relative perturbation bounds for hyperbolic singular values. We then give perturbation bounds for the left and right singular vectors. In Section 3 we consider G with full row rank. We give relative perturbation bounds for hyperbolic singular values and perturbation bounds for the left and right singular vectors. In both sections we compare our bounds with the existing relative perturbation results for the classical SVD. Finally, in Section 4 we discuss some issues which involve the application of the bounds and give three numerical examples.

2 Full column rank case

Let $G \in \mathbf{C}^{m\times n}$ have full column rank, that is, $\operatorname{rank}(G) = n \leq m$, and let $J_{ii} \in \{-1, 1\}$, $i = 1, \ldots, n$. Then the HSDV of the pair (G, J) can be written as

$$G = U \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^{-1}, \tag{3}$$

where $U \in \mathbf{C}^{m\times m}$ is unitary, $V \in \mathbf{C}^{n\times n}$ is J-unitary, $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$ and $\sigma_i > 0$ for $i = 1, \ldots, n$. The corresponding eigenvalue problem (1) can be written as

$$GJG^* = U \begin{bmatrix} \Lambda & \\ & 0 \end{bmatrix} U^*, \qquad \Lambda = \Sigma^2 J, \tag{4}$$

and the corresponding hyperbolic eigenvalue problem (2) can be written as

$$V^*G^*GV = |\Lambda|, \qquad V^*JV = J. \tag{5}$$

Notice that inversion of equation $V^*JV = J$ together with $J^{-1} = J$ implies that $V^{-1}JV^{-*} = J$, so V^{-*} is J-unitary, too. Further, we easily see that the matrix V given by (5), or even more generally, by (2), satisfies

$$\kappa(V) = \|V\|\,\|V^{-1}\| = \|V\|^2. \tag{6}$$

Moreover, all V which perform the diagonalization (5) have the same condition number [17].

Notice that U, Σ and V are related as follows. Let us partition U as

$$U = \begin{bmatrix} U_\Sigma & U_0 \end{bmatrix},$$

where the columns of $U_\Sigma = U_{:,1:n}$ and $U_0 = U_{:,n+1:m}$ span the non-zero and zero subspaces of the matrix GJG^*, respectively. Then

$$U_\Sigma = GV\Sigma^{-1}.$$

As already mentioned, here we will consider G scaled from the right-hand side. Let $G = BD$, where D is $n \times n$ non-singular matrix. For the structured perturbation

$$\delta G = \delta B D \tag{7}$$

we have

$$\widetilde{G} = G + \delta G = (B + \delta B)D = (I + \delta G G^\dagger)G = (I + \delta B B^\dagger)G. \tag{8}$$

For the ease of the presentation, define

$$E = \delta B B^\dagger, \qquad \beta = \|E\| = \|\delta B B^\dagger\|, \qquad \beta_F = \|E\|_F = \|\delta B B^\dagger\|_F. \tag{9}$$

Obviously, if $\|\delta B\|$ or $\|\delta B\|_F$ are known, then

$$\beta \le \frac{\|\delta B\|}{\sigma_{\min}(B)}, \qquad \beta_F \le \frac{\|\delta B\|_F}{\sigma_{\min}(B)}.$$

In particular, for the element-wise perturbation of G of the form

$$|\delta G| \le \varepsilon |G|, \tag{10}$$

we can choose D to be a diagonal matrix, in which case (10) is equivalent to

$$|\delta B| \le \varepsilon |B|.$$

Additionally, if this diagonal D is such that $\|B_{:,i}\| = 1$, $i = 1, \ldots, n$, we have

$$\beta \leq \frac{\| \delta B \|}{\sigma_{\min}(B)} \leq \frac{\| \delta B \|_F}{\sigma_{\min}(B)} = \frac{\| |\delta B| \|_F}{\sigma_{\min}(B)} \leq \varepsilon \frac{\| |B| \|_F}{\sigma_{\min}(B)} = \varepsilon \frac{\sqrt{n}}{\sigma_{\min}(B)}.$$

Notice that both types of structured perturbations, (10) and (7) with D as above, appear in numerical computations. The first type appears when the matrix is being stored in computer memory, and the second type appears in some algorithms during floating-point computations [1, 3, 11]. In particular, one part of the algorithm for highly accurate symmetric eigenreduction from [11] is a one-sided Jacobi-type algorithm for computing the HSVD of the pair (G, J), and the perturbation bounds of this section can be used to analyze that algorithm.

2.1 Singular value bounds

In this section we assume that the unperturbed and the perturbed hyperbolic singular values, σ_i and $\tilde{\sigma}_i$, respectively, are arranged in the increasing order. The relative Weyl-type perturbation bound for hyperbolic singular values follows directly from [19, Theorem 3.3].

Theorem 1 *Let G, $\tilde{G}$, δG and β be as in (3), (8) and (9), respectively. If $\beta < 1$, then the unperturbed and the perturbed hyperbolic singular values σ_i and $\tilde{\sigma}_i$, respectively, satisfy the inequalities*

$$1 - \beta \leq \frac{\tilde{\sigma}_i}{\sigma_i} \leq 1 + \beta.$$

Proof. Let $H = GJG^*$ be as in (4) and let $\widetilde{H} = \tilde{G}J\tilde{G}^*$ be corresponding perturbed matrix. Since

$$\|\delta G x\| = \|\delta B D x\| = \|\delta B B^\dagger B D x\| \leq \|\delta B B^\dagger\| \, \|B D x\| = \beta \, \|G x\|,$$

H and $\widetilde{H}$ fulfill the assumptions of [19, Theorem 3.3], and the theorem follows by taking the square root of the bound of [19, Theorem 3.3]. ∎

The following theorem gives a relative Wielandt–Hoffman-type perturbation bound for hyperbolic singular values.

Theorem 2 *Let G, $\tilde{G}$, δG, E, β and β_F be as in (3), (8) and (9), respectively. If $2\beta_F + \beta_F^2 < 2/3$, then the unperturbed and the perturbed hyperbolic singular values σ_i and $\tilde{\sigma}_i$, respectively, satisfy the inequality*

$$\sqrt{\sum_{i=1}^{n} \left(\frac{\sigma_i^2 - \tilde{\sigma}_i^2}{\sigma_i \tilde{\sigma}_i} \right)^2} \leq \psi \left(\frac{1}{2}\psi + \sqrt{1 + \frac{1}{4}\psi^2} \right),$$

where

$$\psi = \frac{2\beta_F + \beta_F^2}{\sqrt{1 - 2\beta - \beta^2}}. \tag{11}$$

Proof. Let $H' = G^*G$ be as in (5), and let $\widetilde{H}' \equiv H' + \delta H' = \widetilde{G}^*\widetilde{G}$. According to (8)

$$\widetilde{H}' = G^*(I+E)^*(I+E)G = G^*(I+E^*+E+E^*E)G.$$

We can now apply [12, Theorem 3] with $D = G$, $A = I$, and

$$\delta A = E^* + E + E^*E,$$

and the theorem follows. ∎

Since

$$\frac{|\sigma_i^2 - \tilde{\sigma}_i^2|}{\sigma_i\tilde{\sigma}_i} \geq \frac{|\sigma_i - \tilde{\sigma}_i|\,|\sigma_i + \tilde{\sigma}_i|}{(\max\{\sigma_i, \tilde{\sigma}_i\})^2} \geq \frac{|\sigma_i - \tilde{\sigma}_i|}{\max\{\sigma_i, \tilde{\sigma}_i\}},$$

the bound of Theorem 2 further implies

$$\sqrt{\sum_{i=1}^{n}\left(\frac{\sigma_i - \tilde{\sigma}_i}{\max\{\sigma_i, \tilde{\sigma}_i\}}\right)^2} \leq \psi\left(\frac{1}{2}\psi + \sqrt{1 + \frac{1}{4}\psi^2}\right).$$

Notice that the hyperbolic singular values behave as well as the singular values. Namely, the relative perturbations in Theorems 1 and 2 depend on norms of δB and $B^\dagger$ as do the bounds from [3, Theorem 2.14] and [8, Theorem 4.4, Remark 4.2], respectively.

2.2 Singular vector bounds

In this section we present relative variants of the well-known $\sin\Theta$ theorems [2] for left and right singular subspaces of the pair (G, J). Let $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ be two subspaces of the same dimension. The $\sin\Theta$ theorems give the bound for $\|\sin\Theta(\mathcal{U},\widetilde{\mathcal{U}})\|$, where $\sin\Theta(\mathcal{U},\widetilde{\mathcal{U}})$ is diagonal matrix whose diagonal elements are the sines of canonical angles between the subspaces $\mathcal{U}$ and $\widetilde{\mathcal{U}}$. The matrix $\sin\Theta(\mathcal{U},\widetilde{\mathcal{U}})$ is defined as follows [14, Corollary I.5.4]: let $U_\perp$ and $\widetilde{U}$ form orthonormal basis for $\mathcal{U}_\perp$ and $\widetilde{\mathcal{U}}$, respectively, where $\mathcal{U}_\perp$ is the orthogonal complement of $\mathcal{U}$, and let QSW^* be a singular value decomposition of $U_\perp^*\widetilde{U}$. Then $\sin\Theta(\mathcal{U},\widetilde{\mathcal{U}}) = S$.

Throughout this section we assume that the unperturbed and the perturbed hyperbolic singular values are in the same order. More precisely, σ_i denotes the k-th largest hyperbolic singular value of the pair (G, J), and $\tilde{\sigma}_i$ denotes the k-th largest hyperbolic singular value of the perturbed pair $(\widetilde{G}, J)$.

Let the HSVD from (3) be written as

$$G = \begin{bmatrix} U_1 & U_2 & U_0 \end{bmatrix} \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}^{-1}, \tag{12}$$

where $U_1 = U_{:,1:k}$ and the rest of the matrices have the corresponding dimensions. Similarly, let

$$\widetilde{G} = \begin{bmatrix} \widetilde{U}_1 & \widetilde{U}_2 & \widetilde{U}_0 \end{bmatrix} \begin{bmatrix} \widetilde{\Sigma}_1 & \\ & \widetilde{\Sigma}_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{V}_1 & \widetilde{V}_2 \end{bmatrix}^{-1}. \tag{13}$$

Also, let

$$J = J_1 \oplus J_2, \qquad \Lambda = \Lambda_1 \oplus \Lambda_2, \qquad \widetilde{\Lambda} = \widetilde{\Lambda}_1 \oplus \widetilde{\Lambda}_2,$$

be partitioned accordingly. Thus,

$$\Lambda_1 = \Sigma_1^2 J_1, \quad \Lambda_2 = \Sigma_2^2 J_2, \quad \widetilde{\Lambda}_1 = \widetilde{\Sigma}_1^2 J_1, \quad \widetilde{\Lambda}_2 = \widetilde{\Sigma}_2^2 J_2. \tag{14}$$

In order to state and prove our theorems we need to define a relative gap between the singular values from $\widetilde{\Sigma}_1$ and those from Σ_2. We shall use two kinds of relative gap,

$$\mathrm{rg}_1(\widetilde{\Sigma}_1, \Sigma_2) = \min_{\substack{1 \le p \le k \\ k+1 \le q \le n}} \frac{|\widetilde{\sigma}_p^2 J_{pp} - \sigma_q^2 J_{qq}|}{\widetilde{\sigma}_p^2 + \sigma_q^2}, \tag{15}$$

and

$$\mathrm{rg}_2(\widetilde{\Sigma}_1, \Sigma_2) = \min_{\substack{1 \le p \le k \\ k+1 \le q \le n}} \frac{|\widetilde{\sigma}_p^2 J_{pp} - \sigma_q^2 J_{qq}|}{\widetilde{\sigma}_p \sigma_q}. \tag{16}$$

The relative gap rg_1 is based on the relative distance $\rho_1(\lambda, \widetilde{\lambda})$ defined by [9, (2.2)] between the unperturbed and perturbed eigenvalues of the corresponding Hermitian eigenvalue problem (4). Notice that ρ_1 is a metric on $\mathbf{R}$ [9, Appendix B]. The relative gap rg_2 is based on the relative distance $\chi(\lambda, \widetilde{\lambda})$ defined by [9, (2.3)] between the unperturbed and perturbed eigenvalues of the problem (4).

The first theorem contains the perturbation bound for the left (unitary) singular subspace spanned by the first k columns of U. Notice that, since we do not assume any particular order for hyperbolic singular values, the first k columns can correspond to any subset of k singular values. The same remark applies to all subsequent singular vector theorems.

Theorem 3 *Let G and $\widetilde{G}$ be as in (12) and (13), respectively, and let δG, β and β_F be defined by (8) and (9), respectively. Further, let $\mathcal{U}_1$ and $\widetilde{\mathcal{U}}_1$ be the subspaces spanned by the columns of U_1 and $\widetilde{U}_1$, respectively, and let $\mathrm{rg}_1(\widetilde{\Sigma}_1, \Sigma_2)$ be defined by (15). If $\beta < 1$ and $\mathrm{rg}_1(\widetilde{\Sigma}_1, \Sigma_2) > 0$, then*

$$\|\sin\Theta(\mathcal{U}_1, \widetilde{\mathcal{U}}_1)\|_F \le \left(\frac{\beta_F}{1-\beta} + \beta_F \right) \frac{1}{\mathrm{rg}_1(\widetilde{\Sigma}_1, \Sigma_2)}.$$

Proof. Let $H = GJG^*$ be as in (1). By using (8) we can write

$$\widetilde{H} = \widetilde{G}J\widetilde{G}^* = (I+E)GJG^*(I+E)^*, \qquad E = \delta B B^\dagger.$$

Since

$$\widetilde{U}^* \left(\widetilde{H} - H \right) U = \begin{bmatrix} \widetilde{\Lambda} & \\ & 0 \end{bmatrix} \widetilde{U}^* U - \widetilde{U}^* U \begin{bmatrix} \Lambda & \\ & 0 \end{bmatrix},$$

and also

$$\widetilde{U}^* \left(\widetilde{H} - H \right) U = \widetilde{U}^* \left(\widetilde{H}(I+E)^{-*}E + EH \right) U,$$

by using (4) we have

$$\begin{bmatrix} \widetilde{\Lambda} & \\ & 0 \end{bmatrix} \widetilde{U}^* U - \widetilde{U}^* U \begin{bmatrix} \Lambda & \\ & 0 \end{bmatrix} = \begin{bmatrix} \widetilde{\Lambda} & \\ & 0 \end{bmatrix} \widetilde{U}^* (I+E)^{-*} E U + \widetilde{U}^* E U \begin{bmatrix} \Lambda & \\ & 0 \end{bmatrix}. \tag{17}$$

Writing this equality block-wise in the partition from (12), (13) and (14), for the $(1,2)$-block we obtain

$$\widetilde{\Lambda}_1 \widetilde{U}_1^* U_2 - \widetilde{U}_1^* U_2 \Lambda_2 = \widetilde{\Lambda}_1 \widetilde{U}_1^* (I+E)^{-*} E U_2 + \widetilde{U}_1^* E U_2 \Lambda_2.$$

Now, using this component-wise for all pairs of indices (p,q), we have

$$\begin{aligned} |(\widetilde{U}_1^* U_2)_{pq}|\, |(\widetilde{\Lambda}_1)_{pp} - (\Lambda_1)_{qq}| \;\; &\leq \;\; |(\widetilde{\Lambda}_1)_{pp}|\, |(\widetilde{U}_1^*)_{p,:} (I+E)^{-*} E (U_2)_{:,q}| \\ &\quad + |(\widetilde{U}_1^*)_{p,:} E (U_2)_{:,q}|\, |(\Lambda_2)_{qq}|. \end{aligned} \tag{18}$$

Similarly, writing (17) block-wise for the $(1,3)$-block we obtain

$$\widetilde{\Lambda}_1 \widetilde{U}_1^* U_0 = \widetilde{\Lambda}_1 \widetilde{U}_1^* (I+E)^{-*} E U_0.$$

Using this component-wise for all pairs of indices (p,q), we have

$$|(\widetilde{U}_1^* U_0)_{pq}| = |(\widetilde{U}_1^*)_{p,:} (I+E)^{-*} E (U_0)_{:,q}|. \tag{19}$$

Since $\widetilde{U}_1$, U_2 and U_0 are matrices with orthonormal columns, combining (18) and (19), taking norms, and using the definition (15), gives the upper bound

$$\|\widetilde{U}_1^* \, [\, U_2 \quad U_0 \,]\, \|_F \leq \frac{1}{\mathrm{rg}_1(\widetilde{\Sigma}_1, \Sigma_2)} \, (\|(I+E)^{-1} E\|_F + \|E\|_F).$$

Using this,

$$\|\sin \Theta(\mathcal{U}_1, \widetilde{\mathcal{U}}_1)\|_F = \|\widetilde{U}_1^* \, [\, U_2 \quad U_0 \,]\, \|_F,$$

and

$$\|(I+E)^{-1} E\|_F + \|E\|_F \leq \frac{\beta_F}{1-\beta} + \beta_F,$$

completes the proof. ■

The second theorem gives the perturbation bound for the right (hyperbolic) singular subspace spanned by the first k columns of V.

Theorem 4 *Let G and $\widetilde{G}$ be as in (12) and (13), respectively, and let δG, β and β_F be defined by (8) and (9), respectively. Further, let $\mathcal{V}_1$ and $\widetilde{\mathcal{V}}_1$ be the subspaces spanned by the columns of V_1 and $\widetilde{V}_1$, respectively. Also, let ψ and $\mathrm{rg}_2(\widetilde{\Sigma}_1, \Sigma_2)$ be defined by (11) and (15), respectively. If $2\beta + \beta^2 < 1$ and $\mathrm{rg}_2(\widetilde{\Sigma}_1, \Sigma_2) > 0$, then*

$$\|\sin \Theta(\mathcal{V}_1, \widetilde{\mathcal{V}}_1)\|_F \leq \|V\|^2 \left(\frac{1}{2}\psi + \sqrt{1 + \frac{1}{4}\psi^2} \right) \frac{\psi}{\mathrm{rg}_2(\widetilde{\Sigma}_1, \Sigma_2)}.$$

Proof. Since the matrix $V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ simultaneously diagonalizes the pair (G^*G, J), the theorem follows directly from [12, Theorem 4]. ∎

Notice that the left (unitary) singular vectors in the HSVD behave as well as the left singular vectors in the classical SVD. Namely, the bound of Theorem 3 essentially depends on norms of δB and $B^\dagger$ and the relative gap, as do the bounds from [3, Theorem 2.16, Corollary 2.17] and [9, Theorem 4.3].

On the other side, the bound of Theorem 4 for the right (hyperbolic) singular vectors has an additional factor $\|V\|^2$ over the corresponding bounds from [3, Theorem 2.16, Corollary 2.17] and [9, Theorem 4.3]. When applying Theorem 4 to the classical SVD with $J = I$ this term vanishes since V is unitary. According to (6), $\|V\|^2 = \kappa(V)$. However, the spectral condition number of the non-unitary eigenvectors appears naturally in various other absolute and relative perturbation results like [19, Theorem 3.17], [4, 5, 7, 12, 18].

In order to simplify the computation of the bound of Theorem 4, we can further bound $\|V\|$ as follows: according to [13, Theorem 3], V which diagonalizes the pair (G^*G, J) satisfies

$$\|V\|^2 \leq \min_{\Delta} \sqrt{\kappa(\Delta^* G^* G \Delta)},$$

where the minimum is over all matrices which commute with J. By taking $\Delta = D^{-1}$, where $G = BD$, we have

$$\|V\|^2 \leq \sqrt{\kappa(D^{-1} G^* G D^{-1})} = \sqrt{\kappa(B^* B)} = \kappa(B). \tag{20}$$

3 Full row rank case

This case is more complicated then the full column case. As we have mentioned in the introduction, the HSVD exists only if GJG^* is non-singular. Also, all bounds have the factor $\|V\|$, and unlike the full column rank case, this factor can be estimated only in some special cases.

Let $G \in \mathbf{C}^{m\times n}$ have full row rank, $\operatorname{rank}(G) = m \leq n$, and let $J_{ii} \in \{-1, 1\}$, $i = 1, \ldots, n$. The HSVD for the pair (G, J) can be written as

$$G = U \begin{bmatrix} \Sigma & 0 \end{bmatrix} V^{-1}. \tag{21}$$

Here $U \in \mathbf{C}^{m\times m}$ is unitary,

$$V^* J V = \bar{J} \equiv \bar{J}_\Sigma \oplus \bar{J}_0, \tag{22}$$

where $\bar{J}$ is some permutation of J, and $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_m)$, where $\sigma_i > 0$.

The corresponding non-singular eigenvalue problem (1) can be written as

$$GJG^* = U \Lambda U^*, \tag{23}$$

where $\Lambda = \Sigma^2 \bar{J}_\Sigma$. The corresponding hyperbolic eigenvalue problem (2) can be written as

$$V^* G^* G V = \begin{bmatrix} |\Lambda| & \\ & 0 \end{bmatrix}, \qquad V^* J V = \bar{J}. \tag{24}$$

Let us partition V as

$$V = [\, V_\Sigma \quad V_0 \,], \tag{25}$$

where $V_\Sigma = V_{:,1:m}$. Then, for all V which perform the diagonalization (24), the parts V_Σ have the same spectral norm (see [18, Proof of Theorem 1]). Clearly, if V performs the diagonalization (24), then any $\hat{V}$ of the form

$$\hat{V} = V \begin{bmatrix} I & \\ & \hat{V}_0 \end{bmatrix} = [\, V_\Sigma \quad V_0 \hat{V}_0 \,],$$

where $\hat{V}_0^* \bar{J}_0 \hat{V}_0 = \bar{J}_0$, does the same.

As already mentioned in the introduction, here we will consider G scaled from the left-hand side. Let $G = \bar{D}\bar{B}$, where $\bar{D}$ is $m \times m$ non-singular matrix. For the structured perturbation

$$\delta G = \bar{D} \delta \bar{B}$$

we have

$$\widetilde{G} = G + \delta G = G(I + G^\dagger \delta G) = G(I + \bar{B}^\dagger \delta \bar{B}). \tag{26}$$

For the ease of presentation, define

$$\bar{E} = \bar{B}^\dagger \delta \bar{B}, \qquad \bar{\beta} = \|\bar{E}\| = \|\bar{B}^\dagger \delta \bar{B}\|, \qquad \bar{\beta}_F = \|\bar{E}\|_F = \|\bar{B}^\dagger \delta \bar{B}\|_F. \tag{27}$$

Obviously, if $\|\delta B\|$ or $\|\delta B\|_F$ are known, then

$$\bar{\beta} \le \frac{\|\delta \bar{B}\|}{\sigma_{\min}(\bar{B})}, \qquad \bar{\beta}_F \le \frac{\|\delta \bar{B}\|_F}{\sigma_{\min}(\bar{B})}.$$

In particular, for the element-wise perturbation of G of the form

$$|\delta G| \le \varepsilon |G|,$$

we can choose D to be a diagonal matrix, in which case the above inequality is equivalent to

$$|\delta B| \le \varepsilon |B|.$$

Additionally, if this diagonal D is such that $\|B_{i,:}\| = 1$, $i = 1, \ldots, m$, we have

$$\bar{\beta} \le \frac{\|\,|\delta \bar{B}|\,\|}{\sigma_{\min}(\bar{B})} \le \frac{\varepsilon \sqrt{m}}{\sigma_{\min}(\bar{B})}.$$

3.1 Singular value bounds

In this section we assume that the unperturbed and the perturbed hyperbolic singular values, σ_i and $\widetilde{\sigma}_i$, respectively, are arranged in the increasing order. The relative Weyl-type perturbation bound for hyperbolic singular values follows directly from [18, Theorem 1].

Theorem 5 *Let G, $\widetilde{G}$, δG and $\bar{\beta}$ be as in (21), (26) and (27), respectively. If $2\bar{\beta}+\bar{\beta}^2<1$, then the unperturbed and the perturbed hyperbolic singular values σ_i and $\widetilde{\sigma}_i$, respectively, satisfy the inequalities*

$$\sqrt{1-(2\bar{\beta}+\bar{\beta}^2)\|V_\Sigma\|^2} \leq \frac{\widetilde{\sigma}_i}{\sigma_i} \leq \sqrt{1+(2\bar{\beta}+\bar{\beta}^2)\|V_\Sigma\|^2}.$$

Proof. Let $H=GJG^*$ be non-singular as in (23). Since,

$$\|\delta G^* x\| = \|\delta\bar{B}^*\bar{D}x\| = \|\delta\bar{B}^*(\bar{B}^*)^\dagger\bar{B}^*\bar{D}x\| \leq \|\delta\bar{B}^*(\bar{B}^*)^\dagger\|\,\|G^*x\| = \bar{\beta}\,\|G^*x\|,$$

the bound follows by taking the square root of the bound of [18, Theorem 1]. ∎

The following theorem gives a relative Wielandt–Hoffman-type perturbation bound for hyperbolic singular values.

Theorem 6 *Let G, $\widetilde{G}$, δG, $\bar{E}$, $\bar{\beta}$ and $\bar{\beta}_F$ be as in (21), (26) and (27), respectively. Let the HSVD of $\widetilde{G}$ be given by*

$$\widetilde{G} = \widetilde{U}\begin{bmatrix}\widetilde{\Sigma} & 0\end{bmatrix}\widetilde{V}^{-1},$$

and let $\widetilde{V}$ be partitioned according to (25). If $\bar{\beta}<1$, then the unperturbed and the perturbed hyperbolic singular values σ_i and $\widetilde{\sigma}_i$, respectively, satisfy the inequality

$$\sqrt{\sum_{i=1}^{m}\left(\frac{\sigma_i^2-\widetilde{\sigma}_i^2}{\sigma_i\widetilde{\sigma}_i}\right)^2} \leq \|V_\Sigma\|\,\|\widetilde{V}_\Sigma\|\,\frac{2\bar{\beta}_F+\bar{\beta}_F^2}{1-\bar{\beta}},$$

provided that the right-hand side is less than 2.

Proof. Let $H=GJG^*$ and $\widetilde{H}=\widetilde{G}J\widetilde{G}^*$ be the corresponding unperturbed and the perturbed eigenvalue problems (23). Then

$$\widetilde{H} = G(I+\bar{E})J(I+\bar{E})^*G^*.$$

Write $\delta H = \widetilde{H}-H$ as

$$\widetilde{H}-H = \widetilde{G}(J(I+\bar{E})^*-(I+\bar{E})^{-1}J)G^* = \widetilde{G}\,\Xi\,G^*,$$

where

$$\Xi = J(I+\bar{E})^*-(I+\bar{E})^{-1}J.$$

By pre- and post-multiplying this equality by $\widetilde{U}^*$ and U, respectively, and using (23), we obtain

$$\widetilde{U}^*\widetilde{G}\,\Xi\,G^*U = \widetilde{U}^*(\widetilde{H} - H)U = \widetilde{U}^*(\widetilde{U}\widetilde{\Lambda}\widetilde{U}^* - U\Lambda U^*)U = \widetilde{\Lambda}\widetilde{U}^*U - \widetilde{U}^*U\Lambda. \qquad (28)$$

From (21), (22) and (25), we have

$$U^*G = [\,\Sigma \quad 0\,]\,V^{-1} = \Sigma \bar{J}_\Sigma V_\Sigma^* J.$$

Analogous equalities hold for the perturbed problem, as well. Inserting the above equality into (28) gives

$$\widetilde{\Lambda}\widetilde{U}^*U - \widetilde{U}^*U\Lambda = |\widetilde{\Lambda}|^{1/2}\bar{J}_\Sigma \widetilde{V}_\Sigma^* J\,\Xi\,JV_\Sigma \bar{J}_\Sigma |\Lambda|^{1/2}. \qquad (29)$$

Set $S = \widetilde{U}^*U$. By interpreting the above equality component-wise and taking the Frobenius norm we obtain

$$\sum_{i,j=1}^{m}\left(\frac{|\widetilde{\lambda_i} - \lambda_j|}{\sqrt{|\widetilde{\lambda_i}|\,|\lambda_j|}}\right)^2 |S_{ij}|^2 = \|\bar{J}_\Sigma \widetilde{V}_\Sigma^* J\,\Xi\,JV_\Sigma \bar{J}_\Sigma\|_F^2.$$

Define the matrix Y by $Y_{ij} = |S_{ij}|^2$. Since Y is a doubly stochastic matrix (see [6, 8]), by applying [8, Lemma 5.1] we have

$$\sum_{i=1}^{m}\left(\frac{|\widetilde{\lambda_i} - \lambda_{\tau(i)}|}{\sqrt{|\widetilde{\lambda_i}||\lambda_{\tau(i)}|}}\right)^2 \le \|\bar{J}_\Sigma \widetilde{V}_\Sigma^* J\,\Xi\,JV_\Sigma \bar{J}_\Sigma\|_F^2, \qquad (30)$$

for some permutation τ of $\{1, 2, \ldots, m\}$. Since

$$\begin{aligned}\Xi &= (I + \bar{E})^{-1}((I + \bar{E})J(I + \bar{E})^* - J)\\ &= (I + \bar{E})^{-1}(\bar{E}J + J\bar{E}^* + \bar{E}J\bar{E}^*),\end{aligned}$$

(27) implies

$$\|\Xi\|_F \le \frac{2\,\bar{\beta}_F + \bar{\beta}_F^2}{1 - \bar{\beta}}. \qquad (31)$$

Inserting into (30) gives

$$\sqrt{\sum_{i=1}^{m}\left(\frac{|\widetilde{\lambda_i} - \lambda_{\tau(i)}|}{\sqrt{|\widetilde{\lambda_i}||\lambda_{\tau(i)}|}}\right)^2} \le \|V_\Sigma\|\,\|\widetilde{V}_\Sigma\|\,\frac{2\,\bar{\beta}_F + \bar{\beta}_F^2}{1 - \bar{\beta}}. \qquad (32)$$

By assumption, the right-hand side of (32) is less then 2. Thus,

$$\frac{|\widetilde{\lambda_i} - \lambda_{\tau(i)}|}{\sqrt{|\widetilde{\lambda_i}||\lambda_{\tau(i)}|}} \le 2,$$

which implies that the permutation τ does not mix eigenvalues of different signs. Applying [8, Proposition 2.4, (2.12)] separately to the parts of the left hand-side of (32) which contain only positive and only negative eigenvalues, respectively, gives

$$\sqrt{\sum_{i=1}^{m}\left(\frac{|\widetilde{\lambda_i}-\lambda_i|}{\sqrt{|\widetilde{\lambda_i}||\lambda_i|}}\right)^2} \leq \sqrt{\sum_{i=1}^{m}\left(\frac{|\widetilde{\lambda_i}-\lambda_{\tau(i)}|}{\sqrt{|\widetilde{\lambda_i}||\lambda_{\tau(i)}|}}\right)^2},$$

which completes the proof. ■

Notice that the bound of Theorem 6 has a factor $\|V_\Sigma\|\,\|\widetilde{V}_\Sigma\|$ which is essentially the condition number of a diagonalizing non-unitary matrix. Similar factors are also present in other relative perturbation bounds for problems with non-unitary diagonalizing matrices as in [18], [12, Theorem 5], [19, Theorem 3.17] and [4].

Remark 1 The fact that the additional factor $\|V_\Sigma\|\,\|\widetilde{V}_\Sigma\|$ involves the perturbed and the unperturbed quantity is rather inconvenient in applying the bound of Theorem 6. Also, contrary to the full column rank case of Section 2 where we have an upper bound (20), the norm $\|V_\Sigma\|$ which appears in Theorems 5 and 6 can be estimated only in some special cases. In Section 4 we describe how the norm $\|\widetilde{V}\|$ can be bounded by $\|V\|$.

3.2 Singular vector bounds

Now we present the relative $\sin\Theta$ theorems for left and right singular subspaces of the pair (G, J). Let the HSVD from (21) be written as

$$G = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Sigma_2 & 0 \end{bmatrix} \begin{bmatrix} V_1 & V_2 & V_0 \end{bmatrix}^{-1}. \tag{33}$$

where $U_1 = U_{:,1:k}$ and the rest of the matrices have the corresponding dimensions. Similarly, let

$$\widetilde{G} = \begin{bmatrix} \widetilde{U}_1 & \widetilde{U}_2 \end{bmatrix} \begin{bmatrix} \widetilde{\Sigma}_1 & 0 & 0 \\ 0 & \widetilde{\Sigma}_2 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{V}_1 & \widetilde{V}_2 & \widetilde{V}_0 \end{bmatrix}^{-1}. \tag{34}$$

Also, let $\bar{J}$ be partitioned accordingly,

$$\bar{J} = \bar{J}_1 \oplus \bar{J}_2 \oplus \bar{J}_0.$$

Similarly as in Section 2.2, we assume that the unperturbed and the perturbed hyperbolic singular values are in the same order. That is, σ_i and $\widetilde{\sigma}_i$ denote the k-th largest hyperbolic singular values of the pairs (G, J) and $(\widetilde{G}, J)$, respectively.

The first theorem contains the perturbation bound for left (unitary) singular subspace spanned by the first k columns of U.

Theorem 7 *Let G and $\widetilde{G}$ be as in (33) and (34), respectively, and let δG, $\bar{\beta}$ and $\bar{\beta}_F$ be defined by (26) and (27), respectively. Let $\mathcal{U}_1$ and $\widetilde{\mathcal{U}}_1$ be the subspaces spanned by the columns of U_1 and $\widetilde{U}_1$, respectively. Let $\mathrm{rg}_2(\widetilde{\Sigma}_1, \Sigma_2)$ be defined as in (16), but with $k+1 \leq q \leq m$. If $\bar{\beta} < \sqrt{2}-1$ and $\mathrm{rg}_2(\widetilde{\Sigma}_1, \Sigma_2) > 0$, then*

$$\|\sin\Theta(\mathcal{U}_1, \widetilde{\mathcal{U}}_1)\|_F \leq \|\widetilde{V}_1\|\,\|V_2\|\,\frac{2\,\bar{\beta}_F + \bar{\beta}_F^2}{1-\bar{\beta}} \cdot \frac{1}{\mathrm{rg}_2(\widetilde{\Sigma}_1, \Sigma_2)}.$$

Proof. The proof is similar to the proof of [16, Theorem 3]. As in the proof of Theorem 6, (29) holds, where Ξ is bounded by (31). For the $(1,2)$ block of (29) we have

$$\widetilde{\Lambda}_1 \widetilde{U}_1^* U_2 - \widetilde{U}_1^* U_2 \Lambda_2 = |\widetilde{\Lambda}_1|^{1/2} \bar{J}_1 \widetilde{V}_1^* J\, \Xi\, J V_2 \bar{J}_2 |\Lambda_2|^{1/2}.$$

By interpreting this equality component-wise and taking the Frobenius norm, we have

$$\|\widetilde{U}_1^* U_2\|_F \leq \|\widetilde{V}_1\|\,\|V_2\|\,\|\Xi\|_F\,\frac{1}{\mathrm{rg}_2(\widetilde{\Sigma}_1, \Sigma_2)},$$

as desired. ■

Notice that the eigenvector bound of Theorem 7 complements the eigenvalue bounds given in [18]. Also, as already mentioned in Remark 1, in Section 4 we describe how to bound the perturbed quantity $\|\widetilde{V}_1\|$ by the unperturbed one, $\|V\|$, and give bounds for $\|V\|$ in some special cases.

Finally, we present the bound for the right (hyperbolic) singular subspace spanned by the first k columns of the matrix V.

Theorem 8 *Let G and $\widetilde{G}$ be as in (33) and (34), respectively, and let δG, $\bar{\beta}$ and $\bar{\beta}_F$ be defined by (26) and (27), respectively. Let $\mathcal{V}_1$ and $\widetilde{\mathcal{V}}_1$ be the subspaces spanned by the columns of V_1 and $\widetilde{V}_1$, respectively. Let $\mathrm{rg}_1(\widetilde{\Sigma}_1, \Sigma_2)$ be defined as in (15), but with $k+1 \leq q \leq m$. If $\bar{\beta} < 1$ and $\mathrm{rg}_1(\widetilde{\Sigma}_1, \Sigma_2) > 0$, then*

$$\|\sin\Theta(\mathcal{V}_1, \widetilde{\mathcal{V}}_1)\|_F \leq \|V\|^2\,\|\widetilde{V}\|^2 \left(\frac{\bar{\beta}_F}{1-\bar{\beta}} + \bar{\beta}_F\right) \cdot \frac{1}{\mathrm{rg}_1(\widetilde{\Sigma}_1, \Sigma_2)}.$$

Proof. Since the columns of V_1 and $\widetilde{V}_1$ are not orthonormal, in order to apply the definition of canonical angles from Section 2.2, we first have to find the orthogonal basis for the subspaces $\widetilde{\mathcal{V}}_1$ and $(\mathcal{V}_1)_\perp$. We do this by using QR factorization. Let

$$\widetilde{V}_1 = \widetilde{Q}\widetilde{R},$$

be the economical QR factorization of $\widetilde{V}_1$. Then, the columns of $\widetilde{Q}$ form the orthogonal basis for the subspace $\widetilde{\mathcal{V}}_1$. Further, let

$$J\,[\,V_2 \quad V_0\,] \begin{bmatrix} \bar{J}_2 & \\ & \bar{J}_0 \end{bmatrix} = QR,$$

be the respective economical QR factorization. Then, the columns of Q form the orthogonal basis for the subspace $(\mathcal{V}_1)_\perp$. Indeed,

$$Q^*V_1 = R^{-*}\begin{bmatrix} \bar{J}_2 & \\ & \bar{J}_0 \end{bmatrix}\begin{bmatrix} V_2^* \\ V_0^* \end{bmatrix} JV_1 = 0.$$

The definition of canonical angles from Section 2.2 now implies

$$\begin{aligned} \|\sin\Theta(\mathcal{V}_1,\widetilde{\mathcal{V}}_1)\|_F &= \|Q^*\widetilde{Q}\|_F = \|R^{-*}(\bar{J}_2\oplus\bar{J}_0)\,[\,V_2 \quad V_0\,]^*\,J\widetilde{V}_1\widetilde{R}^{-1}\|_F \\ &\leq \|R^{-1}\|\,\|\widetilde{R}^{-1}\|\,\|\,[\,V_2 \quad V_0\,]^*\,J\widetilde{V}_1\|_F. \end{aligned} \tag{35}$$

Since

$$\begin{aligned} \|R^{-1}\| &= \frac{1}{\sigma_{\min}([\,V_2 \quad V_0\,])} \leq \frac{1}{\sigma_{\min}(V)} = \|V^{-1}\| = \|V\|, \\ \|\widetilde{R}^{-1}\| &= \frac{1}{\sigma_{\min}(\widetilde{V}_1)} \leq \frac{1}{\sigma_{\min}(\widetilde{V})} = \|\widetilde{V}^{-1}\| = \|\widetilde{V}\|, \end{aligned} \tag{36}$$

it remains to bound $\|\,[\,V_2 \quad V_0\,]^*\,J\widetilde{V}_1\|_F$.

By using (26), we have

$$\widetilde{G}^*\widetilde{G} = (I+\bar{E})^*G^*G(I+\bar{E}), \qquad \bar{E} = \bar{B}^\dagger\delta\bar{B}.$$

Further,

$$\widetilde{G}^*\widetilde{G} - G^*G = \widetilde{G}^*\widetilde{G}(I+\bar{E})^{-1}\bar{E} + \bar{E}^*G^*G. \tag{37}$$

From (24) we have

$$\begin{aligned} G^*GV &= V^{-*}\begin{bmatrix} |\Lambda| & \\ & 0 \end{bmatrix} = JV\begin{bmatrix} \Lambda & \\ & 0 \end{bmatrix}, \\ \widetilde{V}^*\widetilde{G}^*\widetilde{G} &= \begin{bmatrix} |\widetilde{\Lambda}| & \\ & 0 \end{bmatrix}\widetilde{V}^{-1} = \begin{bmatrix} \widetilde{\Lambda} & \\ & 0 \end{bmatrix}\widetilde{V}^*J. \end{aligned}$$

Pre- and post-multiplying (37) by $\widetilde{V}^*$ and V, respectively, and using the above equalities, gives

$$\begin{aligned} \begin{bmatrix} \widetilde{\Lambda} & \\ & 0 \end{bmatrix}\widetilde{V}^*JV - \widetilde{V}^*JV\begin{bmatrix} \Lambda & \\ & 0 \end{bmatrix} &= \begin{bmatrix} \widetilde{\Lambda} & \\ & 0 \end{bmatrix}\widetilde{V}^*J(I+\bar{E})^{-1}\bar{E}V \\ &\quad + \widetilde{V}^*\bar{E}^*JV\begin{bmatrix} \Lambda & \\ & 0 \end{bmatrix}. \end{aligned} \tag{38}$$

For the $(1,2)$-block in the partition from (33) and (34), we have

$$\widetilde{\Lambda}_1\,\widetilde{V}_1^*JV_2 - \widetilde{V}_1^*JV_2\Lambda_2 = \widetilde{\Lambda}_1\widetilde{V}_1^*J(I+\bar{E})^{-1}\bar{E}V_2 + \widetilde{V}_1^*\bar{E}^*JV_2\Lambda_2. \tag{39}$$

Similarly, for the $(1,3)$-block of (38) we have

$$\widetilde{\Lambda}_1\,\widetilde{V}_1^*JV_0 = \widetilde{\Lambda}_1\widetilde{V}_1^*J(I+\bar{E})^{-1}\bar{E}V_0, \tag{40}$$

Using (39) and (40) component-wise (similarly as in (18) and (19)), taking norms, and using the definition of the relative gap, gives the upper bound

$$\|\widetilde{V}_1^* J \,[\, V_2 \quad V_0 \,]\,\|_F \leq \frac{1}{\mathrm{rg}_1(\widetilde{\Sigma}_1, \Sigma_2)} \|V\| \, \|\widetilde{V}\| \left(\frac{\bar{\beta}_F}{1-\bar{\beta}} + \bar{\beta}_F \right).$$

The theorem now follows by inserting this and (36) into (35). ■

4 Applying the bounds

As we already explained in Remark 1, the bounds of Theorems 6, 7 and 8, contain norms of (parts of) hyperbolic singular vector matrices V and $\widetilde{V}$. These terms are inconvenient since having both perturbed and unperturbed quantities makes it impossible to compute the bound in the case when only norm of perturbation $\bar{\delta} B$ and not the perturbation itself is known.

To solve this problem, we have to bound $\|\widetilde{V}\|$ in terms of $\|V\|$. First notice that if $m < n-1$, then the matrices V and $\widetilde{V}$ from (33) and (34), respectively, are not unique. More precisely, any matrix of the form

$$V' = \left[\begin{array}{ccc} V_1 & V_2 & V_0 V_0' \end{array} \right],$$

where V_0' is a $(n-m)\times(n-m)$ $\bar{J}_0$-unitary matrix, is also a hyperbolic singular vector matrix of the pair (G, J). Similarly, any matrix of the form

$$\widetilde{V}' = \left[\begin{array}{ccc} \widetilde{V}_1 & \widetilde{V}_2 & \widetilde{V}_0 \widetilde{V}_0' \end{array} \right], \tag{41}$$

where $\widetilde{V}_0'$ is a $(n-m)\times(n-m)$ $\bar{J}_0$-unitary matrix, is also a hyperbolic singular vector matrix of the pair $(\widetilde{G}, J)$. The proof of the following theorem, which is long and technical, is similar to the one in [16, Section 3], and is therefore omitted.

Theorem 9 *Let G, $\widetilde{G}$, V and $\widetilde{V}$ be as in (33) and (34), respectively. Let δG, $\bar{\beta}$ and $\bar{\beta}_F$ be defined by (26) and (27), respectively, and set $\gamma = \bar{\beta}_F/(1-\bar{\beta})$. If $0 < 4\gamma\|V\|^2 < 1$, then we can choose a $\bar{J}_0$-unitary matrix $\widetilde{V}_0'$ such that the matrix $\widetilde{V}'$ from (41) satisfies*

$$\|\widetilde{V}'\| \leq \frac{\|V\|}{\sqrt{1-4\gamma\|V\|^2}}.$$

Therefore, by using the above inequality we can bound the terms $\|\widetilde{V}_\Sigma\|$, $\|\widetilde{V}_1\|$ and $\|\widetilde{V}\|$ from Theorems 6, 7 and 8, in terms of $\|V\|$.

As we have already mentioned, the bounds for the HSVD differ from the bounds for the classical SVD mainly in having an additional factor which depends on J-unitary matrix V. Naturally, we would like to know when can we expect this additional factor to be small. Unfortunately, when G has full row rank, we can not, in general, efficiently bound $\kappa(V) \equiv \|V\| \, \|V^{-1}\| = \|V\|^2$, as we have done in (20). However, we can bound V in a satisfactory manner when the associated Hermitian matrix $H = GJG^*$ has some special structure: the bound for the case when H is a *scaled diagonal dominant matrix* is given in [18, (30)]; the bound for the case when H is positive definite is given in [18, Theorem 5]; the bound for the case when H is *quasi definite matrix* is given in [16, Section 3.1]; and the bound for the case when H is *block-scaled diagonally dominant matrix* is given in [15].

Let us mention that when both matrices G and GJG^* are square and non-singular, we can apply bounds from Sections 2 and 3. The question which bounds are then sharper, has no general answer. We can say that the answer depends on whether G is well scaled from the left or from the right. For example, if we can write $G = BD$ with B well-conditioned, than the bounds of Section 2 might be sharper. Vice versa, if $\bar{B}$ from $G = \bar{D}\bar{B}$ is well-conditioned, then the bounds of Section 3 are likely to be better.

We illustrate our results by three computed examples. Our first example illustrates bounds of Section 2.

Example 1 Let $G = \hat{B}\hat{D}$ with

$$\begin{aligned} \hat{B} &= \begin{bmatrix} 0.24 & -0.62 & -0.86 & -0.51 & 0.88 \\ 0.90 & -0.02 & -0.37 & 0.17 & -0.32 \\ 0.28 & -0.18 & 0.22 & 0.01 & -0.20 \\ -0.51 & -0.07 & -0.65 & -0.07 & -0.38 \\ -0.29 & 0.22 & 0.24 & 0.08 & -0.18 \end{bmatrix}, \qquad (42) \\ \hat{D} &= \operatorname{diag}(10^{10}, 100, 1, 1, 10^5), \end{aligned}$$

and let $J = \operatorname{diag}(1, 1, 1, -1, -1)$. The hyperbolic singular values of the pair (G, J) are (properly rounded)

$$\sigma_1^+ = 1.14 \cdot 10^{10}, \; \sigma_2^- = 1.04 \cdot 10^5, \; \sigma_3^+ = 44.5, \; \sigma_4^+ = 0.747, \; \sigma_5^- = 0.0354.$$

Here and in the subsequent examples the superscript "+" ("−") denotes that σ_i corresponds to positive (negative) diagonal entry of J. Notice that $\kappa(G) = 3.2 \cdot 10^{11}$, while choosing a diagonal matrix D such that $\|B_{:,i}\| = 1$, $i = 1, \ldots, n$, gives $G = BD$ with $\kappa(B) = 35.4$. Therefore, the matrix G is well-scaled from the right, thus the bounds of Section 2 are appropriate.

Let the perturbed matrix be given by $\widetilde{G} = (\hat{B} + \delta\hat{B})\hat{D}$ with

$$\delta\hat{B} = 10^{-6} \cdot \begin{bmatrix} 0.46 & -0.26 & -0.02 & 0.36 & 0.42 \\ -0.26 & -0.96 & -0.66 & 0.21 & 0.69 \\ -0.02 & -0.66 & -0.91 & -0.15 & 0.14 \\ 0.36 & 0.21 & -0.15 & 0.27 & -0.33 \\ 0.42 & 0.69 & 0.14 & -0.33 & -0.35 \end{bmatrix}. \qquad (43)$$

This is a relative component-wise perturbation satisfying $|\delta G| \leq 4.8 \cdot 10^{-5} |G|$.

When applying Theorem 2, the exact perturbation of the hyperbolic singular values is equal to $2.44 \cdot 10^{-5}$ and the bound of the theorem is $7.53 \cdot 10^{-5}$. The relative gaps for individual singular values, the true perturbations of the singular vectors corresponding to individual singular values, and the bounds obtained by using Theorems 3 and 4 are displayed in Table 1. Notice that the hyperbolic singular vector matrix V which appears in the bound of Theorem 4 is well conditioned, $\kappa(V) = \|V\|^2 = 1.14$.

i	1	2	3	4	5
$\mathrm{rg}_1(\tilde{\sigma}_i, \Sigma_2)$	1	1	0.99	0.99	1
$\|\sin\Theta(\mathcal{U}_i, \tilde{\mathcal{U}}_i)\|_F$	$6 \cdot 10^{-7}$	$3 \cdot 10^{-7}$	10^{-6}	10^{-6}	$2 \cdot 10^{-6}$
Theorem 3	$8 \cdot 10^{-5}$	$8 \cdot 10^{-5}$	$8 \cdot 10^{-5}$	$8 \cdot 10^{-5}$	$8 \cdot 10^{-5}$
$\mathrm{rg}_2(\tilde{\sigma}_i, \Sigma_2)$	10^5	2339	60	21	21
$\|\sin\Theta(\mathcal{V}_i, \tilde{\mathcal{V}}_i)\|_F$	10^{-11}	$2 \cdot 10^{-10}$	$2 \cdot 10^{-8}$	$3 \cdot 10^{-7}$	$3 \cdot 10^{-7}$
Theorem 4	$8 \cdot 10^{-10}$	$4 \cdot 10^{-8}$	$2 \cdot 10^{-6}$	$4 \cdot 10^{-6}$	$4 \cdot 10^{-6}$

Table 1: Singular vector perturbations for $G = \hat{B}\hat{D}$

Our second example illustrates bounds of Section 3.

Example 2 Let $G = \hat{D}\hat{B}$, where $\hat{B}$ and $\hat{D}$ are as in (42). Let $J = \mathrm{diag}(1, 1, 1, -1, -1)$. The hyperbolic singular values of the pair (G, J) are (properly rounded)

$$\sigma_1^+ = 3.8 \cdot 10^9, \ \sigma_2^- = 3.97 \cdot 10^4, \ \sigma_3^+ = 51.9, \ \sigma_4^+ = 1.02, \ \sigma_5^- = 0.173.$$

Notice that $\kappa(G) = 9.3 \cdot 10^{10}$, while choosing a diagonal matrix $\bar{D}$ such that $\|\bar{B}_{i,:}\| = 1$, $i = 1, \ldots, n$, gives $G = \bar{D}\bar{B}$ with $\kappa(\bar{B}) = 29.4$. Therefore, the matrix G is well-scaled from the left, thus the bounds of Section 3 are appropriate.

Let the perturbed matrix be given by $\tilde{G} = \hat{D}(\hat{B} + \delta\hat{B})$, where $\delta\hat{B}$ is defined by (43). When applying Theorem 6, the exact perturbation of the hyperbolic singular values is equal to $2.22 \cdot 10^{-5}$ and the bound of the theorem is $1.56 \cdot 10^{-3}$. The relative gaps for individual singular values, the true perturbations of the singular vectors corresponding to individual singular values, and the bounds obtained by using Theorems 7 and 8 are displayed in Table 2. For the hyperbolic singular vector matrices we have $\|V\| \approx \|\tilde{V}\| \approx 6.28$.

The two previous examples illustrate the relative perturbation bounds for the HSVD of the pair (G, J) when the matrix G is scaled from the right or from the left, respectively. In our third example we consider the HSVD of the pair (G, J) when the matrix G is scaled from both sides.

i	1	2	3	4	5
$\mathrm{rg}_2(\tilde{\sigma}_i, \Sigma_2)$	$9.7 \cdot 10^4$	765	50.9	6.05	6.05
$\|\sin\Theta(\mathcal{U}_i, \widetilde{\mathcal{U}}_i)\|_F$	10^{-11}	10^{-8}	10^{-7}	$3 \cdot 10^{-6}$	$3 \cdot 10^{-6}$
Theorem 7	$8 \cdot 10^{-9}$	10^{-6}	$9 \cdot 10^{-6}$	$7 \cdot 10^{-5}$	$4 \cdot 10^{-5}$
$\mathrm{rg}_1(\tilde{\sigma}_i, \Sigma_2)$	1	1	0.99	0.99	1
$\|\sin\Theta(\mathcal{V}_i, \widetilde{\mathcal{V}}_i)\|_F$	$5 \cdot 10^{-7}$	$8 \cdot 10^{-7}$	$7 \cdot 10^{-6}$	$5 \cdot 10^{-6}$	$5 \cdot 10^{-6}$
Theorem 8	$8 \cdot 2^{-3}$	$2 \cdot 10^{-3}$	$4 \cdot 2^{-3}$	$2 \cdot 10^{-3}$	$2 \cdot 10^{-3}$

Table 2: Singular vector perturbations for $G = \hat{D}\hat{B}$

Example 3 Let $G = \hat{D}_1\hat{B}\hat{D}$, where $\hat{B}$ and $\hat{D}$ are as in (42) and

$$\hat{D}_1 = \mathrm{diag}(10^4, 10^{-2}, 10^3, 1, 100).$$

Let $J = \mathrm{diag}(1,1,1,-1,-1)$ The hyperbolic singular values of the pair (G, J) are (properly rounded)

$$\sigma_1^+ = 3.7 \cdot 10^{11},\ \sigma_2^- = 7.98 \cdot 10^6,\ \sigma_3^+ = 1.56 \cdot 10^2,\ \sigma_4^+ = 1.26,\ \sigma_5^- = 0.0241.$$

Notice that $\kappa(G) = 1.5 \cdot 10^{13}$, while choosing a diagonal matrix D such that $\|\bar{B}_{:,i}\| = 1$, $i = 1, \ldots, n$, gives $G = BD$ with $\kappa(B) = 6.2 \cdot 10^3$. Therefore, we can use the bounds of Section 2.

Let the perturbed matrix be given by $\widetilde{G} = \hat{D}_1(\hat{B} + \delta\hat{B})\hat{D}$, where $\delta\hat{B}$ is defined by (43). When applying Theorem 2, the exact perturbation of the hyperbolic singular values is equal to $3.56 \cdot 10^{-5}$ and the bound of the theorem is $4.78 \cdot 10^{-3}$. The relative gaps for individual singular values, the true perturbations of the singular vectors corresponding to individual singular values, and the bounds obtained by using Theorems 3 and 4 are displayed in Table 3.

On the other hand, here we can not apply the bounds of Section 3. Namely, setting $G = \bar{D}\bar{B}$ and $\delta G = \bar{D}\delta\bar{B}$ for any non-singular diagonal matrix $\bar{D}$, the quantities $\bar{\beta}$ and $\bar{\beta}_F$ from (27) are both approximately equal to $1.3 \cdot 10^5$. This is due to the fact that, although G is scaled from both sides, the scaling from the right is dominant.

Acknowledgements

We would like to thank the referee for very careful reading of the manuscript and valuable comments.

References

[1] J. Barlow and J. Demmel, Computing accurate eigensystems of scaled diagonally dominant matrices, *SIAM J. Numer. Anal.*, 27:762–791 (1990).

i	1	2	3	4	5
$\mathrm{rg}_1(\tilde{\sigma}_i, \Sigma_2)$	1	1	1	1	1
$\\|\sin\Theta(\mathcal{U}_i, \widetilde{\mathcal{U}}_i)\\|_F$	10^{-6}	10^{-6}	$2\cdot 10^{-6}$	$2\cdot 10^{-6}$	$3\cdot 10^{-7}$
Theorem 3	$5\cdot 10^{-3}$	$5\cdot 10^{-3}$	$5\cdot 10^{-3}$	$5\cdot 10^{-3}$	$5\cdot 10^{-3}$
$\mathrm{rg}_2(\tilde{\sigma}_i, \Sigma_2)$	$5\cdot 10^{4}$	$5\cdot 10^{4}$	124	52	52
$\\|\sin\Theta(\mathcal{V}_i, \widetilde{\mathcal{V}}_i)\\|_F$	$2\cdot 10^{-11}$	$9\cdot 10^{-10}$	$6\cdot 10^{-8}$	$3\cdot 10^{-6}$	$3\cdot 10^{-6}$
Theorem 4	10^{-7}	10^{-7}	$5\cdot 10^{-5}$	10^{-4}	10^{-4}

Table 3: Singular vector perturbations for $G = \hat{D}_1\hat{B}\hat{D}$

[2] C. Davis and W. M. Kahan, The rotation of eigenvectors by a perturbation. III, *SIAM J. Numer. Anal.*, 7:1–46 (1970).

[3] J. Demmel and K. Veselić, Jacobi's method is more accurate than QR, *SIAM J. Matrix Anal. Appl.*, 13:1204–1244 (1992).

[4] S. C. Eisenstat and I. C. F. Ipsen, Three absolute perturbation bounds for matrix eigenvalues imply relative bounds, *SIAM J. Matrix Anal. Appl.*, 20:149–158 (1998).

[5] S. C. Eisenstat and I. C. F. Ipsen, Relative perturbation results for eigenvalues and eigenvectors of diagonalisable matrices, *BIT*, 38:502–509 (1998).

[6] L. Elsner and S. Friedland, Singular values, doubly stochastic matrices, and applications, *Linear Algebra Appl.*, 220:161–169 (1995).

[7] I. C. F. Ipsen, Absolute and relative perturbation bounds for invariant subspaces of matrices, *Linear Algebra Appl.*, 309:45–56 (2000).

[8] R.-C. Li, Relative perturbation theory: (i) eigenvalue and singular value variations, *SIAM J. Matrix Anal. Appl.*, 19:956–982 (1998).

[9] R.-C. Li, Relative perturbation theory: (ii) eigenspace and singular subspace variations, *SIAM J. Matrix Anal. Appl.*, 20:471–492 (1998).

[10] R. Onn, A. O. Steinhardt and A. Bojanczyk, Hyperbolic singular value decomposition and applications, *IEEE Transactions on Acoustics, Speech, and Signal Processing*, 1575–1588 (1991).

[11] I. Slapničar, Accurate Symmetric Eigenreduction by a Jacobi Method, Ph. D. thesis, Fernuniversität Hagen, Hagen, Germany, 1992.

[12] I. Slapničar and N. Truhar, Relative perturbation theory for hyperbolic eigenvalue problem, *Linear Algebra Appl.*, 309:57–72 (2000).

[13] I. Slapničar and K.Veselić, A bound for the condition of a hyperbolic eigenvector matrix, *Linear Algebra Appl.* 290:247–255 (1999).

[14] G. W. Stewart and J.-G. Sun, *Matrix Perturbation Theory*, Academic Press, Boston, 1990.

[15] N. Truhar and R.-C. Li, A $\sin 2\Theta$ Theorem for Graded Indefinite Hermitian Matrices, Tehnical Report 2000-33, Department of Mathematics, University of Kentucky, submitted to Linear Algebra Appl.

[16] N. Truhar and I. Slapničar, Relative perturbation bounds for invariant subspaces of graded indefinite Hermitian matrices, *Linear Algebra Appl.* 301:171–185 (1999).

[17] K. Veselić, A Jacobi eigenreduction algorithm for definite matrix pairs, *Numerische Math.*, 64:241–269 (1993).

[18] K. Veselić, Perturbation theory for the eigenvalues of factorised symmetric matrices, *Linear Algebra Appl.*, 309:85–102 (2000).

[19] K. Veselić and I. Slapničar, Floating-point perturbations of Hermitian matrices, *Linear Algebra Appl.*, 195:81–116 (1993).

[20] H. Zha, A note on the existence of the hyperbolic singular value decomposition, *Linear Algebra Appl.*, 240:199-205 (1996).