A Bound for the Condition of a Hyperbolic Eigenvector Matrix

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Abstract

The hyperbolic eigenvector matrix is a matrix X which simultaneously diagonalizes the pair (H, J), where H is Hermitian positive definite and $J = \mathrm{diag}(\pm 1)$ such that $X^*HX = \Delta$ and $X^*JX = J$. We prove that the spectral condition of X, $\kappa(X)$, is bounded by $\kappa(X) \leq \sqrt{\min \kappa(D^*HD)}$, where the minimum is taken over all non-singular matrices D which commute with J. This bound is attainable and it can be simply computed. Similar results hold for other signature matrices J, like in the discretized Klein-Gordon equation.

1 Introduction

We are considering the hyperbolic eigenvalue problem

$$H\mathbf{x} = \lambda J\mathbf{x}, \tag{1}$$

where H is a $n \times n$ Hermitian positive definite matrix, and $J = \mathrm{diag}(\pm 1)$. There always exists a matrix X such that

$$X^*HX = \Delta, \qquad X^*JX = J, \tag{2}$$

where Δ is diagonal positive definite matrix. Since H is positive definite, the pair (H, J) is regular by definition from [9, Definition VI.1.2], so the existence of X follows from [9, Theorem VI.1.15] and [9, Corollary VI.1.19]. The matrix X is also called J-unitary. Obviously, the i-th eigenvalue of the problem (1) is given by

$$\lambda_i = \Delta_{ii} J_{ii},$$

and the i-th column of X is the corresponding eigenvector. We call such eigenvectors hyperbolic, or J-unitary, contrary to the standard unitary eigenvectors of the problem $H\mathbf{x} = \lambda\mathbf{x}$. The matrix X is also called a hyper-exchange matrix with respect to the signature matrix J [5].

The matrix X also appears in other linear algebra problems. For example, X is the eigenvector matrix of the matrix JH,

$$X^{-1}(JH)X = JX^*J(JH)X = J\Delta.$$

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Also, X is right singular vector matrix of the *hyperbolic singular value decomposition* (HSVD) of the pair (G, J). The HSVD for the full column-rank G is defined as

$$G = U\Sigma X^*,$$

where

$$U^*U = I, \quad X^*JX = J, \quad \Sigma = \operatorname{diag}(\sigma_i), \quad \sigma_i > 0.$$

Such HSVD is used in the highly accurate algorithm for the eigenvalue decomposition of a possibly indefinite symmetric (Hermitian) matrix A [11, 7]: the idea is to factorize A as $A = GJG^*$ [8] and then compute the HSVD of the pair (G, J). Further, HSVD and its variant for the full row-rank G is a suitable way to compute the eigenvalue decomposition of the difference of two outer products [15, 5], and the condition of X appears in the perturbation bounds for the eigenvalues of the non-singular matrix GJG^* [13]. Also, note that hyperbolic eigenvalue problems with other signature matrices (c.f. section 3) arise within some Lanczos-type algorithms for non-symmetric matrices [4].

In the paper $\|\cdot\|$ denotes the the spectral matrix norm, and $\kappa(A)$ denotes the condition of a non-singular matrix A,

$$\kappa(A) = \|A\| \, \|A^{-1}\|.$$

The hyperbolic eigenvector matrix has two important properties:

1. All matrices which perform the simultaneous diagonalization (2) have the same condition [11].
2. $\kappa(X) = \|X\|^2$. Moreover, the singular values of X come in pairs of reciprocals, $\{\sigma, 1/\sigma\}$.

The condition $\kappa(X)$ can be expressed in terms of a Hermitian matrix which is associated to the problem (1). Let us define the spectral absolute value $|A|_S$ of the Hermitian matrix A as its positive definite polar factor. That is, if $A = Q\Lambda Q^*$ is the eigenvalue decomposition of A, then

$$|A|_S = Q|\Lambda|Q^* = \sqrt{A^2}.$$

Theorem 1 *Let $H = Z^*Z$ be some factorization of H. Then*

$$\kappa(X) = \max_{\mathbf{x}\neq 0} \frac{\mathbf{x}ZZ^*\mathbf{x}}{\mathbf{x}^*|ZJZ^*|_S\mathbf{x}} = \max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^*|ZJZ^*|_S\mathbf{x}}{\mathbf{x}ZZ^*\mathbf{x}}.$$

Proof. The first equality was proved in [13], and the second equality follows because the eigenvalues of XX^* come in the pairs of reciprocals. ∎

Note that the spectral absolute value appears naturally in the relative perturbation bounds for Hermitian and normal matrices [14, 1].

Since the maxima in Theorem 1 are not easy to compute, it is of interest to obtain a simpler bound for $\kappa(X)$. Veselić [12] recently proved that

$$\kappa(X) \leq \min_{D\in\mathcal{D}} \kappa(D^*HD),$$

where $\mathcal{D}$ is the set of all non-singular matrices which commute with J. In this paper we shall prove a better bound, namely

$$\kappa(X) \leq \sqrt{\min_{D\in\mathcal{D}} \kappa(D^*HD)}. \tag{3}$$

We shall also show for which matrices D the minimum is attained, and for which matrices H the bound itself is attained.

The rest of the paper is organized as follows: in section 2 we prove the above results, and in section 3 we apply our results to eigenvalue problems with other signature matrices, and in particular to the discretized Klein-Gordon equation and some Hamiltonian systems.

2 Bound for $\kappa(X)$

We shall prove the bound (3) for $\kappa(X)$ in two stages: we shall first analyze the case when the bound is an equality, and then prove the bound itself. From now on we assume without loss of generality that J has the form

$$J = \begin{bmatrix} I_m & 0 \\ 0 & -I_{n-m} \end{bmatrix}, \tag{4}$$

which is easily achieved by permutation. Since all results of this section are trivial if in $m = 0$ or $m = n$, we assume that $0 < m < n$. Also,

$$\mathcal{D} = \{D = D_1 \oplus D_2 : D_1 \in \mathbf{C}^{m,m}, D_2 \in \mathbf{C}^{n-m,n-m}, \text{ nonsingular}\}.$$

will denote the set of all non-singular matrices which commute with J from (4). To prove our results we need the following theorem which appeared in [3] (see also [2]).

Theorem 2 *[3, Theorem 2] Let*

$$H = \begin{bmatrix} I_m & \Psi \\ \Psi^* & I_{n-m} \end{bmatrix} \tag{5}$$

be positive definite. Then

$$\kappa(H) = \min_{D \in \mathcal{D}} \kappa(D^* H D).$$

The following theorem shows that the bound (3) becomes an equality for matrices of the form (5).

Theorem 3 *Let J and H be given by (4) and (5), respectively. Let X be some matrix which diagonalizes the pair (H, J) according to (2). Then*

$$\kappa(X) = \sqrt{\kappa(H)} = \sqrt{\min_{D \in \mathcal{D}} \kappa(D^* H D)}.$$

Proof. The second equality follows from Theorem 2. Let us construct one particular X. Let $U^* \Psi V = \Sigma = \operatorname{diag}(\sigma_i)$ be the singular value decomposition of Ψ. Set

$$W = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix},$$

and $H_1 = W^* H W$. Then $W^* J W = J$ and

$$H_1 = \begin{bmatrix} I_m & \Sigma \\ \Sigma^T & I_{n-m} \end{bmatrix}.$$

Since H is positive definite we have $\sigma_i \leq \|\Psi\| < 1$, and

$$\kappa(H_1) = \kappa(H) = \frac{1 + \sigma_{\max}}{1 - \sigma_{\max}}.$$

Let R be the matrix which diagonalizes the pair (H_1, J) according to (2), and let $k = \min\{m, n-m\}$. Then

$$R = \begin{bmatrix} c_1 & & & & & s_1 & & & & \\ & c_2 & & & & & s_2 & & & \\ & & \ddots & & & & & \ddots & & \\ & & & c_k & & & & & s_k & \\ & & & & I_{m-k} & & & & & 0 \\ s_1 & & & & & c_1 & & & & \\ & s_2 & & & & & c_2 & & & \\ & & \ddots & & & & & \ddots & & \\ & & & s_k & & & & & c_k & \\ & & & & 0 & & & & & I_{n-m-k} \end{bmatrix} \tag{6}$$

Here, c_i and s_i are hyperbolic sines and cosines computed as follows: if $\sigma_i = 0$, then $c_i = 1$, $s_i = 0$; otherwise

$$\begin{aligned} t_i &= -\frac{\sigma_i}{1+\sqrt{1-\sigma_i^2}}, \\ c_i &= \frac{1}{\sqrt{1-t_i^2}}, \\ s_i &= c_i \cdot t_i. \end{aligned}$$

Now we have $X = WR$, and, since W is unitary, $\kappa(X) = \kappa(R)$. A straightforward computation shows that

$$\kappa(R) = \frac{1+\max|t_i|}{1-\min|t_i|} = \sqrt{\frac{1+\sigma_{\max}}{1-\sigma_{\max}}} = \sqrt{\kappa(H)}.$$

The theorem now follows from the fact that all X which perform the required diagonalization have the same condition. ■

The aim of our main theorem is twofold: to prove the bound (3) and to define the matrix D for which the minimum is attained.

Theorem 4 *Let J be given by (4) and let*

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix} \tag{7}$$

be partitioned accordingly. Let

$$D = \begin{bmatrix} G_1^{-1} & 0 \\ 0 & G_2^{-1} \end{bmatrix},$$

where $H_{ii} = G_i^* G_i$ is some factorization of H_{ii} for $i \in \{1,2\}$, respectively. Let $\widehat{H} = D^* H D$ and let X be the matrix which diagonalizes the pair (H, J) according to (2). Then

$$\kappa(X) \leq \sqrt{\kappa(\widehat{H})} = \sqrt{\min_{D\in\mathcal{D}} \kappa(D^* H D)}. \tag{8}$$

Proof. The equality in (8) follows from Theorem 3 since $\widehat{H}$ has the form (5) with $\Psi = G_1^{-*} H_{12} G_2^{-1}$. Since all X which perform the diagonalization (2) have the same condition, it remains to prove the inequality in (8) for one particular X which we shall now construct.

Let $\widehat{X} = WR$ be the matrix which diagonalizes the pair $(\widehat{H}, J)$ as in the proof of Theorem 3, and let $k = \min\{m, n-m\}$. Then

$$\widehat{X}^* \widehat{H} \widehat{X} = \mathrm{diag} = S^2 = T^2 \oplus I_{m-k} \oplus T^2 \oplus I_{n-m-k},$$

where $T^2 = \mathrm{diag}(1 + \sigma_i \cdot t_i)$. Special forms of R from (6) and S imply that they commute. Set

$$Z = DWRS^{-1}. \tag{9}$$

Since $Z^* H Z = I$ we conclude that the eigenvalue decomposition of the matrix $Z^* J Z$ is given by

$$Z^* J Z = Q J \Delta^{-1} Q^*, \tag{10}$$

where Q is unitary, and Δ is given by (2). Therefore, the matrix

$$X = ZQ\Delta^{1/2}$$

performs the required diagonalization of the pair (H, J). Since $\kappa(X) = \|X\|^2$ we have

$$\kappa(X) = \lambda_{\max}(ZQ\Delta Q^* Z^*). \tag{11}$$

Inverting (10) gives

$$Z^{-1} J Z^{-*} = Q\Delta J Q^* = Q\Delta Q^* Q J Q^*,$$

and inserting this expression for $Q\Delta Q^*$ into (11) gives

$$\kappa(X) = \lambda_{\max}(ZZ^{-1}JZ^{-*}QJQ^*Z^*) = \lambda_{\max}(QJQ^*Z^*JZ^{-*}). \tag{12}$$

In the last equality we have used the fact that for any square matrices A and B, the matrices AB and BA have the same eigenvalues. ¿From (9), by using J-orthogonality of R, the fact that R and S commute, and the fact that D, W and S commute with J, we have

$$Z^* J Z^{-*} = S^{-1} R^* W^* D^* J D^{-*} W^{-*} R^{-*} S = R^* J R^{-*} = R^2 J.$$

By inserting this into (12), by using the fact that $\lambda_i(A) \leq \|A\|$ for any matrix A, and by using unitarity of Q and J, Theorem 3 finally gives

$$\kappa(X) = \lambda_{\max}(QJQ^*R^2J) \leq \|QJQ^*R^2J\| = \|R^2\| = \kappa(R) = \kappa(\widehat{X}) = \sqrt{\kappa(\widehat{H})},$$

as desired. ■

3 Other signature matrices

Theorem 4 yields as a corollary similar results for the eigenvalue problems with other Hermitian and skew-Hermitian signature matrices. Let us consider the simultaneous diagonalization of the pair (H, J_S) where H is positive definite matrix and the signature matrix is given by

$$J_S = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \qquad \text{or} \qquad J_S = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

In both cases we seek X such that $X^* J_S X = J_S$ and

$$X^* H X = \begin{bmatrix} \Delta & \Delta_1 \\ \Delta_1 & \Delta \end{bmatrix} \qquad \text{or} \qquad X^* H X = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}, \tag{13}$$

respectively, where Δ and Δ_1 are diagonal matrices. These forms readily contain the eigenvalues of the problem

$$H\mathbf{x} = \lambda J_S \mathbf{x}, \tag{14}$$

and keep real arithmetic whenever possible. Let us set

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & -I \\ I & I \end{bmatrix} \qquad \text{or} \qquad U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & iI \\ iI & I \end{bmatrix},$$

respectively, where i is the imaginary unit. Then we have

$$U^* J_S U = J \qquad \text{or} \qquad U^* J_S U = iJ,$$

respectively, where $J = I \oplus (-I)$. Now set $\widetilde{H} = U^* H U$, and let $\widetilde{X}$ be the matrix which diagonalizes the pair $(\widetilde{H}, J)$ as in (2) such that $\widetilde{X}^* J \widetilde{X} = J$ and

$$\widetilde{X}^* \widetilde{H} \widetilde{X} = \begin{bmatrix} \Delta + \Delta_1 & 0 \\ 0 & \Delta - \Delta_1 \end{bmatrix} \qquad \text{or} \qquad \widetilde{X}^* \widetilde{H} \widetilde{X} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix},$$

respectively. Then $X = U \widetilde{X} U^*$ performs the diagonalization (13), and, by applying Theorem 4 to the above reduction, we have

$$\kappa(X) = \kappa(\widetilde{X}) \leq \sqrt{\min_{D \in \widetilde{\mathcal{D}}} \kappa(D^* H D)}, \tag{15}$$

where $\widetilde{\mathcal{D}}$ is the set of all non-singular matrices which commute with the respective J_S.

The reduction (13) for the first choice of J_S appears in the case of the discretized Klein-Gordon equation (c.f. [10]) which consists of the quadratic eigenvalue problem

$$(\lambda^2 - 2\lambda V + V^2 - Z^2)\psi = 0,$$

where Z and V are real symmetric matrices, Z is positive definite, and $\|VZ^{-1}\| < 1$. This eigenvalue problem is equivalent to the problem (14) with

$$H = \begin{bmatrix} L^* L & L^{-1} V L \\ L^* V L^{-*} & L^* L \end{bmatrix}, \qquad J_S = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

where $Z = LL^*$ is some factorization of Z. By taking

$$D_0 = \begin{bmatrix} L^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix}$$

which commutes with J_S, the relation (15) implies that the condition of the matrix X which performs the required diagonalization of the pair (H, J_S) is bounded by

$$\kappa(X) \leq \sqrt{\kappa(D_0^* H D_0)} = \sqrt{\frac{1 + \|VZ^{-1}\|}{1 - \|VZ^{-1}\|}}.$$

The reduction (13) for the second choice of J_S comes in solution of certain Hamiltonian systems. It is also part of the highly accurate eigenvalue decomposition algorithm for skew-symmetric matrices [6]. If H is partitioned according to this J_S as in (7), then the minimum in (15) is attained for

$$D = \begin{bmatrix} G_1 + G_2 & -i(G_1 - G_2) \\ i(G_1 - G_2) & G_1 + G_2 \end{bmatrix},$$

where

$$
\begin{aligned}
G_1^* G_1 &= \frac{1}{2} \times [H_{11} + H_{22} + i(H_{12} - H_{12}^*)], \\
G_2^* G_2 &= \frac{1}{2} \times [H_{11} + H_{22} + i(H_{12}^* - H_{12})],
\end{aligned}
$$

and (15) is an equality if $H_{11} + H_{22} = 2I$ and $H_{12} = H_{12}^*$.

Similar bounds can be easily derived for a matrix X which diagonalizes any pair (H, J), where H is positive definite and J satisfies merely the condition

$$
J = J^* = J^{-1} \qquad \text{or} \qquad J = -J^* = -J^{-1}.
$$

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