Relative Perturbation Theory for Hyperbolic Eigenvalue Problem *

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Abstract

We give relative perturbation bounds for eigenvalues and perturbation bounds for eigenspaces of a hyperbolic eigenvalue problem $H\mathbf{x} = \lambda J\mathbf{x}$, where H is positive definite matrix and J is a diagonal matrix of signs. We consider two types of perturbations: when a graded matrix $H = D^*AD$ is perturbed in a graded sense to $H + \delta H = D^*(A + \delta A)D$, and the multiplicative perturbations of the form $H + \delta H = (I+E)^*H(I+E)$. Our bounds are simple to compute, compare well to the classical results, and can be used when analyzing numerical algorithms.

1 Introduction

We are considering the hyperbolic eigenvalue problem

$$H\mathbf{x}_i = \lambda_i J\mathbf{x}_i, \qquad i = 1, 2, \cdots, n, \tag{1}$$

where H is a $n \times n$ Hermitian positive definite matrix, and $J = \mathrm{diag}(\pm 1)$. Since H is positive definite, the pair (H, J) is regular by [19, Definition VI.1.2]. From [19, Theorem VI.1.15] and [19, Corollary VI.1.19], it follows that there always exists a non-singular matrix X such that

$$X^*HX = |\Lambda|, \qquad X^*JX = J, \tag{2}$$

where $|\Lambda|$ is a diagonal positive definite matrix. The i-th eigenvalue of the problem (1) is then given by $\lambda_i \equiv |\Lambda_{ii}|J_{ii}$, and the i-th column of X is the

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corresponding eigenvector. We call such eigenvectors hyperbolic, or J-unitary, contrary to the standard unitary eigenvectors. The matrix X is also called J-unitary[1].

The matrix X appears in some other linear algebra problems. For example, X is the eigenvector matrix of the matrix JH,

$$X^{-1}(JH)X = JX^*J(JH)X = \Lambda.$$

Also, X is the right singular vector matrix of a hyperbolic singular value decomposition (HSVD) of the pair (G, J). The HSVD for the full column-rank G is defined as

$$G = U\Sigma X^*, \tag{3}$$

where

$$U^*U = I, \quad X^*JX = J, \quad \Sigma = \mathrm{diag}(\sigma_i), \quad \sigma_i > 0.$$

Such an HSVD is used in the highly accurate algorithm for the Hermitian eigenvalue decomposition of a possibly indefinite symmetric (Hermitian) matrix A [21, 15]. Further, the HSVD and its variant for the full row-rank G is a suitable way to compute the eigenvalue decomposition of the difference of two outer products [24, 14]. The condition number of X appears in other relative perturbation results [20, 22].

Relative perturbation bounds for eigenvalue problem have been the topic of many articles in past years, such as [1, 3, 23, 10, 11, 4, 16, 12, 13, 5] (see also the review article [8]). Some of the recent works include [9, 20, 22]. These works covered positive definite, indefinite, and diagonalizable matrices. In this paper we give relative perturbation bounds for eigenvalues and perturbation bounds for eigenvectors of the problem (1) under additive and multiplicative perturbations of H. The bounds for eigenvectors are given in terms of eigenspaces. An additive perturbation is defined as follows: H is given in a graded form, $H = D^*AD$, and the perturbed matrix is defined by

$$\widetilde{H} \equiv H + \delta H = D^*(A + \delta A)D.$$

Here we assume that $\widetilde{H}$ is positive definite, as well. As we shall see in Section 3.3, some types of (relative) additive perturbations typically occur in numerical computations. The multiplicative perturbation is defined by a congruence

$$\widetilde{H} = H + \delta H = (I + E)^*H(I + E).$$

For both types of perturbation, the perturbed problem shall be denoted by

$$\widetilde{X}^*\widetilde{H}\widetilde{X} = |\widetilde{\Lambda}|, \qquad \widetilde{X}^*J\widetilde{X} = J. \tag{4}$$

[1] In [14], the matrix X is also called a hyper-exchange matrix with respect to the signature matrix J.

Our bounds are simpler and, in appropriate cases, better than the classical norm-wise estimates [19, Section VI.3] (see Section 3.3).

Throughout the paper we assume that the unperturbed and the perturbed eigenvalues are in the same order. Also, $\|\cdot\|$ denotes the the spectral matrix norm, $\|\cdot\|_F$ denotes the Frobenius norm, and $\kappa(A) = \|A\|\,\|A^{-1}\|$ denotes the spectral condition number.

The rest of the paper is organized as follows. In Section 2 we describe some properties of J-unitary matrices. We give bounds for additive perturbations in Section 3, and for multiplicative perturbations in Section 4.

2 Some properties of J-unitary matrices

First, note that all matrices which perform the diagonalization (2) have the same condition number [21]. Further, it can be easily verified that $\kappa(X) = \|X\|^2$. Moreover, the singular values of X come in pairs of reciprocals, $\{\sigma, 1/\sigma\}$. The following theorem was proved in [17].

Theorem 1 (Slapničar and Veselić) *Let X perform the diagonalization (2). Then*

$$\|X\|^2 = \kappa(X) \leq \sqrt{\min \kappa(F^*HF)},$$

where the minimum is taken over all non-singular matrices F which commute with J.

We shall also need the following lemma.

Lemma 1 *Let J be given by*

$$J = \begin{bmatrix} I_l & \\ & -I_{n-l} \end{bmatrix}. \tag{5}$$

Let X and $\widetilde{X}$ be two J-unitary matrices which are partitioned accordingly in block columns as

$$X = [\, X_p \quad X_n \,] \qquad \textit{and} \qquad \widetilde{X} = [\, \widetilde{X}_p \quad \widetilde{X}_n \,],$$

where

$$X_p^* J X_p = \widetilde{X}_p^* J \widetilde{X}_p = I_l, \qquad X_n^* J X_n = \widetilde{X}_n^* J \widetilde{X}_n = -I_{n-l}.$$

We say that X_p (X_n) spans the positive (negative) subspace with respect to J. Then the matrix $X^* J \widetilde{X}$ is also J-unitary, and we have

$$\|X^* J \widetilde{X}\| = \|X_n^* J \widetilde{X}_p\| + \sqrt{1 + \|X_n^* J \widetilde{X}_p\|^2}\ , \tag{6}$$

$$\|\widetilde{X}\| \leq \left(\|X_n^* J \widetilde{X}_p\| + \sqrt{1 + \|X_n^* J \widetilde{X}_p\|^2} \right) \|X\|. \tag{7}$$

Proof. The equality (6) follows from the CS decomposition of a J unitary matrix $X^* J\widetilde{X}$ (the proof of such CS decomposition is very similar to the proof of [19, Theorem I.5.1]). The inequality (7) follows from (6) since $\|\widetilde{X}\| \leq \|X^{-*}\| \, \|X^* J\widetilde{X}\| = \|X\| \, \|X^* J\widetilde{X}\|$. ∎

3 Bounds for additive perturbations

In this section we derive relative perturbation bounds for eigenvalues and perturbation bound for eigenspaces. We also show that the bounds are applicable in numerical computations.

3.1 Eigenvalue bounds

Let us first analyze the perturbations of the eigenvalues. [23, Theorem 2.1] implies the following bound.

Theorem 2 *Let the perturbation δH satisfy $|\mathbf{x}^* \delta H \mathbf{x}| \leq \eta \, \mathbf{x}^* H \mathbf{x}$ for all $\mathbf{x} \in \mathbf{C}^n$ and some $\eta < 1$. Then the pairs (H, J) and $(\widetilde{H}, J)$ have the same inertia, and*

$$\frac{|\widetilde{\lambda}_i - \lambda_i|}{|\lambda_i|} \leq \eta. \tag{8}$$

Note that $\log_{10}(|\widetilde{\lambda}_i - \lambda_i|/|\lambda_i|)$ can be used to measure the number of correct decimal digits in numerical results. For relative additive perturbations we have the following corollary.

Corollary 1 *Let $H = D^* A D$ and $\widetilde{H} = D^*(A + \delta A)D$. Then (8) holds with $\eta = \|\delta A\| \, \|A^{-1}\|$, provided that $\eta < 1$.*

Proof. We have

$$|\mathbf{x}^* \delta H \mathbf{x}| = \|\mathbf{x}^* D^* \delta A D \mathbf{x}\| \leq \|\delta A\| \, \|A^{-1}\| \, \|\mathbf{x}^* D^* A D \mathbf{x}\|. \qquad ∎$$

Now we shall establish Hofmann–Wielandt type bound. We need two auxiliary lemmas on doubly stochastic matrices. We say that a real $n \times n$ matrix Y is doubly stochastic if $Y_{ij} \geq 0$ and $\sum_{k=1}^n Y_{ik} = \sum_{k=1}^n Y_{ki} = 1$ for $k = 1, 2, \cdots, n$. By Birkhoff's theorem [7, Theorem 8.7.1], a matrix is doubly stochastic if and only if it lies in the convex hull of all permutation matrices. This result has lead to the following lemma [10, Lemma 5.1].

Lemma 2 (Li) *Let Y be a $n \times n$ doubly stochastic matrix, and let M be a $n \times n$ complex matrix. Then there exists a permutation τ of $\{1, 2, \cdots, n\}$ such that*

$$\sum_{i,j=1}^n |M_{ij}|^2 Y_{ij} \geq \sum_{i=1}^n |M_{i\tau(i)}|^2.$$

We also need the following lemma from [6].

Lemma 3 (Elsner and Friedland) *For any square matrix M there exists a doubly stochastic matrix Y of the same dimension such that $\sigma_{\min}^2(M)Y_{ij} \leq |M_{ij}|^2$.*

Now we are ready to prove our theorem.

Theorem 3 *Let $H = D^*AD$ and $\widetilde{H} = D^*(A+\delta A)D$, and let the matrices X and $\widetilde{X}$ perform the diagonalizations (2) and (4), respectively. If $\|A^{-1}\delta A\|_F < 2/3$, then*

$$\sqrt{\sum_{i=1}^{n}\left(\frac{\Lambda_{ii}-\widetilde{\Lambda}_{ii}}{\sqrt{\Lambda_{ii}\widetilde{\Lambda}_{ii}}}\right)^2} \leq \psi\left(\frac{1}{2}\psi + \sqrt{1+\frac{1}{4}\psi^2}\right), \tag{9}$$

where

$$\psi = \frac{\|A^{-1}\delta A\|_F}{\sqrt{1-\|A^{-1}\delta A\|}}. \tag{10}$$

Proof. The relation (4) implies

$$(H+\delta H)\widetilde{X} = \widetilde{H}\widetilde{X} = J\widetilde{X}\widetilde{\Lambda}.$$

Premultiply this equality by X^*, use (2), and rearrange, to get

$$\Lambda X^* J\widetilde{X} - X^* J\widetilde{X}\widetilde{\Lambda} = -X^*\delta H\widetilde{X}. \tag{11}$$

Now

$$\begin{aligned} X^*\delta H\widetilde{X} &= X^*D^*\delta AD\widetilde{X} \\ &= X^*D^*A^{1/2}A^{-1/2}\delta AA^{-1/2}\Phi^{-1/2}\Phi^{1/2}A^{1/2}D\widetilde{X}, \end{aligned} \tag{12}$$

where $\Phi = I + A^{-1/2}\delta AA^{-1/2}$. Set $G = A^{1/2}D$ such that $H = G^*G$. By combining (2) with the definition of the HSVD from (3), we see that there exists an unitary matrix U such that $GX = U^*|\Lambda|^{1/2}$. Similarly, there exists an unitary matrix $\widetilde{U}$ such that $\widetilde{G}\widetilde{X} = \widetilde{U}^*|\widetilde{\Lambda}|^{1/2}$, where $\widetilde{G} = \Phi^{1/2}A^{1/2}D$. Therefore, we can rewrite (12) as

$$X^*\delta H\widetilde{X} = |\Lambda|^{1/2}\Psi|\widetilde{\Lambda}|^{1/2},$$

where

$$\Psi = U^*A^{-1/2}\delta AA^{-1/2}\Phi^{-1/2}\widetilde{U}.$$

Thus, (11) becomes

$$\Lambda X^*J\widetilde{X} - X^*J\widetilde{X}\widetilde{\Lambda} = -|\Lambda|^{1/2}\Psi|\widetilde{\Lambda}|^{1/2}. \tag{13}$$

By using this equality component-wise for all pairs of indices (i,j), squaring each term, and adding them all together, we obtain

$$\sum_{i,j=1}^{n}\left(\frac{\Lambda_{ii}-\widetilde{\Lambda}_{jj}}{\sqrt{|\Lambda_{ii}\widetilde{\Lambda}_{jj}|}}\right)^2 |\,[X^*J\widetilde{X}]_{ij}|^2 = \|\Psi\|_F^2.$$

By applying Lemma 3 to the matrix $X^*J\widetilde{X}$, we have

$$\sum_{i,j=1}^{n}\left(\frac{\Lambda_{ii}-\widetilde{\Lambda}_{jj}}{\sqrt{|\Lambda_{ii}\widetilde{\Lambda}_{jj}|}}\right)^2 Y_{ij} \le \|\Psi\|_F^2\|(X^*J\widetilde{X})^{-1}\|^2,$$

for some doubly stochastic matrix Y. Note that, since the matrix $X^*J\widetilde{X}$ is J-unitary, we have $\|(X^*J\widetilde{X})^{-1}\| = \|X^*J\widetilde{X}\|$. Lemma 2 further implies

$$\sum_{i=1}^{n}\left(\frac{\Lambda_{ii}-\widetilde{\Lambda}_{\tau(i)\tau(i)}}{\sqrt{|\Lambda_{ii}\widetilde{\Lambda}_{\tau(i)\tau(i)}|}}\right)^2 \le \|\Psi\|_F^2\|X^*J\widetilde{X}\|^2, \tag{14}$$

for some permutation τ of $\{1,2,\cdots,n\}$.

Let us bound the norms on the right hand side. First, the assumption of the theorem implies that

$$\|\Psi\|_F \le \|A^{-1/2}\delta AA^{-1/2}\|_F\|\Phi^{-1/2}\| \le \frac{\|A^{-1/2}\delta AA^{-1/2}\|_F}{\sqrt{1-\|A^{-1/2}\delta AA^{-1/2}\|}} \le \psi, \tag{15}$$

where ψ is defined by (10). The last inequality follows since all matrices involved are Hermitian. Further, assume J has the form (5), which can be achieved by permutation without loss of generality. By (6) it remains to bound $\|X_n^*J\widetilde{X}_p\|$, where X_n and $\widetilde{X}_p$ are defined in Lemma 1. Use (13) component-wise for all pairs of indices (i,j), where $i = l+1,\cdots,n$ and $j = 1,\cdots,l$, to get

$$[X^*J\widetilde{X}]_{ij} = -\Psi_{ij}\frac{|\Lambda_{ii}|^{1/2}|\widetilde{\Lambda}_{jj}|^{1/2}}{\Lambda_{ii}-\widetilde{\Lambda}_{jj}}.$$

For this particular choice of (i,j), the eigenvalues Λ_{ii} and $\widetilde{\Lambda}_{jj}$ have opposite signs, which implies

$$|\,[X^*J\widetilde{X}]_{ij}| \le \frac{1}{2}|\Psi_{ij}|.$$

Here we have used the fact that $\sqrt{ab}/(a+b) \le 1/2$ for any two positive numbers a and b. By squaring the above inequality and adding all terms together for $i = l+1,\cdots,n$ and $j = 1,\cdots,l$, we obtain

$$\|X_n^*J\widetilde{X}_p\|_F \le \frac{1}{2}\|\Psi\|_F. \tag{16}$$

By using this, (6) and (15), we have

$$\|X^* J \widetilde{X}\| \leq \frac{1}{2}\psi + \sqrt{1 + \frac{1}{4}\psi^2}. \tag{17}$$

Note that the assumption of the theorem, (17) and (15) imply that the right hand side of (14) is not larger than 4. This, in turn, implies that the permutation τ must be such that the quotients in (14) contain only eigenvalues of the same sign (otherwise at lest two terms would be at least 4). By applying [10, Proposition 2.4] separately to quotients which contain only positive and only negative eigenvalues in (14), we conclude that the sum is minimal when τ is identity. Finally, (9) follows by inserting this, (15) and (17) into (14). ■

3.2 Eigenvector bound

Let us partition the eigenvalue problem (2) as

$$\begin{aligned}
\begin{bmatrix} X_1^* \\ X_2^* \end{bmatrix} H \begin{bmatrix} X_1 & X_2 \end{bmatrix} &= \begin{bmatrix} |\Lambda_1| & \\ & |\Lambda_2| \end{bmatrix}, \\
\begin{bmatrix} X_1^* \\ X_2^* \end{bmatrix} J \begin{bmatrix} X_1 & X_2 \end{bmatrix} &= \begin{bmatrix} J_1 & \\ & J_2 \end{bmatrix}.
\end{aligned} \tag{18}$$

By [19, Section VI.2.4], columns of X_1 span a simple eigenspace of the pair (H, J) if Λ_1 and Λ_2 have no common eigenvalues. Let us partition the perturbed problem (4) in the same manner,

$$\begin{aligned}
\begin{bmatrix} \widetilde{X}_1^* \\ \widetilde{X}_2^* \end{bmatrix} \widetilde{H} \begin{bmatrix} \widetilde{X}_1 & \widetilde{X}_2 \end{bmatrix} &= \begin{bmatrix} |\widetilde{\Lambda}_1| & \\ & |\widetilde{\Lambda}_2| \end{bmatrix}, \\
\begin{bmatrix} \widetilde{X}_1^* \\ \widetilde{X}_2^* \end{bmatrix} J \begin{bmatrix} \widetilde{X}_1 & \widetilde{X}_2 \end{bmatrix} &= \begin{bmatrix} J_1 & \\ & J_2 \end{bmatrix}.
\end{aligned} \tag{19}$$

Let

$$V = X^{-1} = \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix}, \tag{20}$$

and let

$$\widetilde{X}_1 = \widetilde{Q}_1 \widetilde{R}_1, \qquad V_2 = Q_2 R_2, \tag{21}$$

be the economical QR factorizations of $\widetilde{X}_1$ and V_2, respectively. By $\mathcal{R}(W)$ we denote the subspace which is spanned by the columns of some matrix W, and by $\mathcal{R}(W)^\perp$ we denote its orthogonal complement. The columns of $\widetilde{Q}_1$ and Q_2 form the orthogonal bases for $\mathcal{R}(\widetilde{X}_1)$ and $\mathcal{R}(X_1)^\perp$, respectively. Let $U\Sigma V^*$ be a singular value decomposition of the matrix $Q_2^* \widetilde{Q}_1$. The diagonal entries of the matrix $\sin\Theta(X_1, \widetilde{X}_1) \equiv \Sigma$ are the sines of canonical angles between the subspaces $\mathcal{R}(X_1)$ and $\mathcal{R}(\widetilde{X}_1)$ [19, Corollary I.5.4]. Our theorem bounds

$\|\sin\Theta(X_1,\widetilde{X}_1)\|_F$, and is, therefore, a variant of the well-known $\sin\Theta$ theorems [2, Section 2], [19, Section V.3] designed for relative perturbations.

Let us define the relative gap between the sets of eigenvalues from $\widetilde{\Lambda}_1$ and Λ_2 as

$$\mathrm{rg}(\widetilde{\Lambda}_1,\Lambda_2)=\min_{i,j}\frac{|\,[\widetilde{\Lambda}_1]_{ii}-[\Lambda_2]_{jj}|}{\sqrt{|\,[\widetilde{\Lambda}_1]_{ii}\cdot[\Lambda_2]_{jj}|}}. \tag{22}$$

This definition is based on a definition of a relative distance $\mathrm{RelDist}(\lambda,\widetilde{\lambda})=|\lambda-\widetilde{\lambda}|/\sqrt{|\lambda\widetilde{\lambda}|}$, which was used in [1], [3] and [10, 11]. In the positive definite case (when $J=I$), all λ and $\widetilde{\lambda}$ are positive and $\mathrm{RelDist}(\lambda,\widetilde{\lambda})$ is a generalized metric [10, Proposition 2.4]. In the hyperbolic case, however, $\mathrm{RelDist}(\lambda,\widetilde{\lambda})$ does not necessarily increase with the distance between λ and $\widetilde{\lambda}$ if they have different signs. For example, if $\Lambda_1=\{1\}$ and $\widetilde{\Lambda}_2=\{-1,0.1\}$, then the minimum is attained between 1 and -1 and not between 1 and 0.1.

Now we can state our theorem.

Theorem 4 *Let $H=D^*AD$ and $\widetilde{H}=D^*(A+\delta A)D$, and let the matrices X and $\widetilde{X}$ perform the diagonalizations (18) and (19), respectively. If $\|A^{-1}\delta A\|<1$, then*

$$\|\sin\Theta(X_1,\widetilde{X}_1)\|_F\le\|X\|^2\left(\frac{1}{2}\psi+\sqrt{1+\frac{1}{4}\psi^2}\right)\frac{\psi}{\mathrm{rg}(\widetilde{\Lambda}_1,\Lambda_2)}, \tag{23}$$

where ψ is defined by (10).

Proof. Let (20) and (21) hold. Then we have

$$\begin{aligned}\|\sin\Theta(X_1,\widetilde{X}_1)\|_F &= \|Q_2^*\widetilde{Q}_1\|_F=\|R_2^{-*}V_2^*\widetilde{X}_1\widetilde{R}_1^{-1}\|_F\\ &\le \|R_2^{-1}\|\,\|R_1^{-1}\|\,\|V_2^*\widetilde{X}_1\|_F\\ &\le \|R_2^{-1}\|\,\|R_1^{-1}\|\,\|X_2^*J\widetilde{X}_1\|_F. \end{aligned}\tag{24}$$

The last inequality follows from (20) and the J-unitarity of X. By interpreting the relation (13) block-wise, we get

$$\Lambda_2X_2^*J\widetilde{X}_1-X_2^*J\widetilde{X}_1\widetilde{\Lambda}_1=-|\Lambda_2|^{1/2}\Psi_{21}|\widetilde{\Lambda}_1|^{1/2}, \tag{25}$$

where

$$\Psi\equiv\begin{bmatrix}\Psi_{11} & \Psi_{12}\\ \Psi_{21} & \Psi_{22}\end{bmatrix}$$

is partitioned according to (18). By interpreting (25) component-wise and using (22), we get

$$|\,[X_2^*J\widetilde{X}_1]_{ji}|\le\frac{|\,[\Psi_{12}]_{ji}|}{\mathrm{rg}(\widetilde{\Lambda}_1,\Lambda_2)},$$

or

$$\|X_2^* J \widetilde{X}_1\|_F \leq \frac{\|\Psi\|_F}{\mathrm{rg}(\widetilde{\Lambda}_1, \Lambda_2)} \leq \frac{\psi}{\mathrm{rg}(\widetilde{\Lambda}_1, \Lambda_2)}. \tag{26}$$

Here the last inequality follows from (15). Further,

$$\|R_2^{-1}\| = \frac{1}{\sigma_{\min}(V_2)} \leq \frac{1}{\sigma_{\min}(V)} = \|X\|. \tag{27}$$

Similarly,

$$\|\widetilde{R}_1^{-1}\| \leq \|\widetilde{X}^{-1}\| = \|\widetilde{X}\|. \tag{28}$$

We can further bound $\|\widetilde{X}\|$ by (7), which, together with (16) and (15), implies

$$\|\widetilde{R}_1^{-1}\| \leq \|X\| \left(\frac{1}{2}\psi + \sqrt{1 + \frac{1}{4}\psi^2} \right).$$

Finally, (23) follows by inserting this, (27) and (26) into (24). ■

An inconvenience of Theorem 4 is that the bound (23) uses both original and perturbed eigenvalues in the definition of relative gap. To compute $\mathrm{rg}(\widetilde{\Lambda}_1, \Lambda_2)$ one thus needs to know the exact perturbation δA. However, in some important cases, like the ones described in Section 3.3, only information about $\|\delta A\|$ is given. It is therefore desirable to have the bound which uses only original eigenvalues. By using Corollary 1, one can easily bound $\mathrm{rg}(\widetilde{\Lambda}_1, \Lambda_2)$ from below by

$$\mathrm{rg}(\widetilde{\Lambda}_1, \Lambda_2) \geq \min_{i,j} \frac{|\, [\Lambda_1]_{ii}\{1 + \eta \cdot \mathrm{sign}([\Lambda_2]_{jj} - [\Lambda_1]_{ii}) \cdot \mathrm{sign}([\Lambda_1]_{ii})\} - [\Lambda_2]_{jj} \,|}{\sqrt{|\, [\Lambda_1]_{ii} [\Lambda_2]_{jj} (1 + \eta) \,|}}.$$

Another inconvenience of Theorem 4 is that the bound (23) contains the term $\|X\|^2$, which is computationally demanding. However, in the important case when the matrix D is diagonal (c.f. Section 3.3), we can use Theorem 1 to bound $\|X\|^2$, thus obtaining the following result.

Corollary 2 *Let $H = DAD$ and $\widetilde{H} = D(A + \delta A)D$, where D is diagonal positive definite matrix, and let the matrices X and $\widetilde{X}$ perform the diagonalizations (18) and (19), respectively. If $\|A^{-1}\delta A\| < 1$, then*

$$\|\sin\Theta(X_1, \widetilde{X}_1)\|_F \leq \sqrt{\kappa(A)} \left(\frac{1}{2}\psi + \sqrt{1 + \frac{1}{4}\psi^2} \right) \frac{\psi}{\mathrm{rg}(\widetilde{\Lambda}_1, \Lambda_2)}.$$

3.3 Applying the bounds

The results of Sections 3.1 and 3.2 enable us to handle two important classes of perturbations which occur in numerical computation. The first class are the perturbations of the form $|\delta H|_{ij} \leq \varepsilon |H_{ij}|$. Such perturbations occur when the matrix H is stored in a computer with machine precision ε. Then, $\|\delta A\| \leq \varepsilon \| \, |A| \, \|$. If we chose $D = \mathrm{diag}(\sqrt{H_{ii}})$, then further $\|\delta A\| \leq \varepsilon n$. Such choice of D is nearly optimal in the sense that $\kappa(A) \leq n \min_\Delta \kappa(\Delta H \Delta)$, where the minimum is over all non-singular diagonal matrices [18]. The second, more general, class of perturbations has the form $|\delta H|_{ij} \leq \varepsilon \sqrt{H_{ii} H_{jj}}$. By choosing the same D as above, we again have $\|\delta A\| \leq \varepsilon n$. Such perturbations occur during various numerical algorithms. Also, for the above perturbations, if $\kappa(A) \ll \kappa(H)$, then our eigenvalue bounds are better than the classical norm-wise bounds like [19, Corollary VI.3.3] which can be applied to the definite matrix pair (H, J). Even more, since the above choice of D is almost optimal, our eigenvalue bounds can never be much worse than the classical bounds. If $\kappa(A) \ll \kappa(H)$, then our eigenvector bound can also be better than the classical bounds like [19, Theorem VI.3.9], in particular when the relative gap is large and the absolute gap is small. This typically occurs in the presence of several tiny eigenvalues.

The following numerical example illustrates some of the above facts.

Example 1 Let $H = DAD$ with

$$A = \begin{bmatrix} 1 & 0.8 & 0.8 & 0.8 & 0.8 \\ 0.8 & 1 & 0.8 & 0.8 & 0.8 \\ 0.8 & 0.8 & 1 & 0.8 & 0.8 \\ 0.8 & 0.8 & 0.8 & 1 & 0.8 \\ 0.8 & 0.8 & 0.8 & 0.8 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 10^{10} & & & & \\ & 50 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 10^5 \end{bmatrix},$$

and let $J = \mathrm{diag}(1, 1, 1, -1, -1)$. The eigenvalues of the pair (H, J) are (properly rounded)

$$\lambda_1 = 10^{20}, \ \lambda_2 = 722, \ \lambda_3 = 0.254, \ \lambda_4 = -0.254, \ \lambda_5 = -3.6 \cdot 10^7.$$

Note that $\kappa(H) = 5 \cdot 10^{20}$ while $\kappa(A) = 21$. Let the perturbed matrix $\widetilde{H}$ be given as in Theorem 3 with

$$\delta A = \begin{bmatrix} 0.46 & -0.26 & -0.02 & 0.36 & 0.42 \\ -0.26 & -0.96 & -0.66 & 0.21 & 0.69 \\ -0.02 & -0.66 & -0.91 & -0.15 & 0.14 \\ 0.36 & 0.21 & -0.15 & 0.27 & -0.33 \\ 0.42 & 0.69 & 0.14 & -0.33 & -0.35 \end{bmatrix}.$$

This is relative component-wise perturbation satisfying $|\delta A| \leq 10^{-6}|A|$. Also, $\|\delta A\| \leq 2 \cdot 10^{-6}$, $\|\delta A\|_F \leq 2.3 \cdot 10^{-6}$. When applying Theorem 3, the left hand side of (9) (the exact perturbation) is equal to $8.4 \cdot 10^{-6}$ and the bound on the right hand side is $1.4 \cdot 10^{-5}$ with

$$\psi \leq \frac{\|A^{-1}\| \, \|\delta A\|_F}{\sqrt{1 - \|A^{-1}\| \, \|\delta A\|}} \approx 1.4 \cdot 10^{-5}.$$

Note that the right hand side of (9) remains the same for any diagonal grading matrix D, thus the bound of Theorem 3 is independent of grading. Also, note that we do not need to know the perturbation δA exactly, since the bound remains approximately the same for any matrix δA which satisfies $|\delta A| \leq 10^{-6}|A|$ or $\|\delta A\| \leq 10^{-6}$. On the other hand, the bounds of [19, Corollary VI.3.3] are completely useless, since there $\|E\| = 0$ but $\|F\| = \|\delta H\| = 4.6 \cdot 10^{13}$ and $\gamma(A, B) = 0.97$ according to [19, Definition VI.1.17].

Let us now consider invariant subspaces. The bound of Corollary 2 is obtained by simply multiplying the eigenvalue bound by $\sqrt{\kappa(A)} = 4.59$ and dividing it by relative gap. Let $\sin \Theta_i = \sin \Theta(\mathbf{x}_i, \widetilde{\mathbf{x}}_i)$, where $\mathbf{x}_i$ is the eigenvector of λ_i. Table 1 gives the corresponding relative gaps, the exact perturbations and the bounds computed by Corollary 2.

i	1	2	3	4	5
$\mathrm{rg}(\widetilde{\lambda}_i, \Lambda_2)$	$1.6 \cdot 10^{6}$	53	2	2	223
$\|\sin \Theta_i\|_F$	$5 \cdot 10^{-14}$	$5 \cdot 10^{-8}$	10^{-7}	10^{-7}	$2 \cdot 10^{-8}$
Corollary 2	$3 \cdot 10^{-11}$	$9 \cdot 10^{-7}$	$3 \cdot 10^{-5}$	$3 \cdot 10^{-5}$	$2 \cdot 10^{-7}$

Table 1: Eigenvector perturbations

Our results can be readily used in the error analysis of some numerical algorithms, in particular the J-orthogonal Jacobi method for solving the problem (1) from [21] and [15, Section 3.1]. One step of this method in floating-point arithmetic with machine precision ε induces the backward error δA such that $\|\delta A\| \leq \mathrm{const} \cdot \varepsilon$ [15, Theorem 3.2.1]. Theorem 3 and Corollary 2 then give bounds for errors in eigenvalues and invariant subspaces after one step of the method. By adding these errors together, similarly as in [15, Corollary 3.2.2], we obtain overall error bounds for the method.

4 Bounds for multiplicative perturbations

In this section we shall give a relative Hofmann–Wielandt type bound, similar to the one of Theorem 3, for the case of multiplicative perturbation of a matrix H. We shall also prove a $\sin \Theta$ type bound, similar to the one of Theorem 4.

Let us first prove our eigenvalue result.

Theorem 5 *Let $H + \delta H = (I+E)^* H (I+E)$, and let the matrices X and $\widetilde{X}$ perform the diagonalizations (2) and (4), respectively. Let*

$$\alpha = \frac{\|E\|_F}{1 - \|E\|}, \qquad \beta = \frac{2\alpha \|X\|^2}{\sqrt{1 - 4\alpha \|X\|^2}}. \tag{29}$$

If $\alpha < 1/(4\|X\|^2)$, then

$$\sqrt{\sum_{i=1}^{n} \left(\frac{|\Lambda_{ii} - \widetilde{\Lambda}_{\tau(i)\tau(i)}|}{\max\{|\Lambda_{ii}|, |\widetilde{\Lambda}_{\tau(i)\tau(i)}|\}} \right)^2} \leq \beta \left(\beta + \sqrt{1+\beta^2} \right), \tag{30}$$

for some permutation τ of $\{1, 2, \cdots, n\}$.

Proof. As in the proof of Theorem 3, (2) and (4) imply (11). Set

$$\widehat{E} = E^* (I+E)^{-*},$$

and note that

$$\|\widehat{E}\|_F \leq \alpha, \qquad \|E\|_F \leq \alpha.$$

Insert $\delta H \equiv \widehat{E}\widetilde{H} + HE$ into (11), and use (2) and (4) to get

$$\Lambda X^* J \widetilde{X} - X^* J \widetilde{X} \widetilde{\Lambda} = -X^* \widehat{E} J \widetilde{X} \widetilde{\Lambda} - \Lambda X^* J E \widetilde{X}.$$

By interpreting this relation component-wise, for each pair (i,j) we get

$$\frac{|\Lambda_{ii} - \widetilde{\Lambda}_{jj}|}{\max\{|\Lambda_{ii}|, |\widetilde{\Lambda}_{jj}|\}} |X^* J \widetilde{X}|_{ij} \leq |X^* \widehat{E} J \widetilde{X}|_{ij} + |X^* J E \widetilde{X}|_{ij}. \tag{31}$$

By squaring each inequality, adding them together, and using Lemma 3 and Lemma 2 as in the proof of Theorem 3, we obtain

$$\begin{aligned} \sqrt{\sum_{i=1}^{n} \left(\frac{|\Lambda_{ii} - \widetilde{\Lambda}_{\tau(i)\tau(i)}|}{\max\{|\Lambda_{ii}|, |\widetilde{\Lambda}_{\tau(i)\tau(i)}|\}} \right)^2} \quad &\leq \quad \|X^* J \widetilde{X}\| \left(\|X^* \widehat{E} J \widetilde{X}\|_F + \|X^* J E \widetilde{X}\|_F \right) \\ &\leq \quad \|X^* J \widetilde{X}\| \, \|X\| \, \|\widetilde{X}\| \, 2\alpha. \end{aligned} \tag{32}$$

It remains to bound $\|X^* J \widetilde{X}\|$ and $\|\widetilde{X}\|$ by using (6) and (7), respectively. To do so we need to bound $\|X_n^* J \widetilde{X}_p\|$, where X_n and $\widetilde{X}_p$ are defined in Lemma 1. For those indices (i,j) which correspond to X_n and $\widetilde{X}_p$, Λ_{ii} is negative and $\widetilde{\Lambda}_{jj}$ is positive, thus the fraction on the left hand side of (31) is always greater than or equal to one. Therefore, (31) implies

$$\|X_n^* J \widetilde{X}_p\|_F \leq \|X_n^* \widehat{E} J \widetilde{X}_p\|_F + \|X^* J E \widetilde{X}\|_F \leq 2\alpha \, \|X\| \, \|\widetilde{X}\|. \tag{33}$$

This and (7) imply

$$\|\widetilde{X}\| - 2\alpha\,\|X\|^2\,\|\widetilde{X}\| \leq \|X\|\sqrt{1+(2\alpha\,\|X\|\,\|\widetilde{X}\|)^2}.$$

The assumption of the theorem implies the positivity of the left hand side. After squaring and rearranging this inequality, we get

$$\|\widetilde{X}\|^2 \leq \frac{\|X\|^2}{1-4\alpha\,\|X\|^2}. \tag{34}$$

By inserting (33) and (34) into (6) we get

$$\|X^* J \widetilde{X}\| \leq \beta + \sqrt{1+\beta^2},$$

where β is defined by (29). Finally, (30) follows by inserting this and (34) into (32). ∎

One difference between additive perturbations (Theorem 3) and this theorem is that the bound (30) depends on the eigenvector matrix X. However, if we set $H = DAD$ where $D = \operatorname{diag}(\sqrt{H_{ii}})$, then Theorem 1 implies $\|X\|^2 \leq \sqrt{\kappa(A)}$, and (30) holds with

$$\beta = \frac{2\alpha\,\sqrt{\kappa(A)}}{\sqrt{1-4\alpha\sqrt{\kappa(A)}}}.$$

Also note that if $\|E\|$, and then, in turn, β, are sufficiently small, then the right hand side of (30) is asymptotically equal to β.

We end the paper by proving an eigenvector bound for multiplicative perturbations.

Theorem 6 *Let $H+\delta H = (I+E)^* H (I+E)$, and let the matrices X and $\widetilde{X}$ perform the diagonalizations (18) and (19), respectively. Let the relative gap between the sets of eigenvalues from $\widetilde{\Lambda}_1$ and Λ_2 be defined as*

$$\overline{\operatorname{rg}}(\widetilde{\Lambda}_1,\Lambda_2) = \min_{i,j} \frac{|\,[\widetilde{\Lambda}_1]_{ii} - [\Lambda_2]_{jj}\,|}{\max\{|\,[\widetilde{\Lambda}_1]_{ii}|, |\,[\Lambda_2]_{jj}|\}}. \tag{35}$$

Let α be defined by (29). If $\alpha < 1/(4\|X\|^2)$, then

$$\|\sin\Theta(X_1,\widetilde{X}_1)\|_F \leq \frac{\|X\|^4}{1-4\alpha\,\|X\|^2}\cdot\frac{2\alpha}{\overline{\operatorname{rg}}(\widetilde{\Lambda}_1,\Lambda_2)}. \tag{36}$$

Further, if we write $H = DAD$, where $D = \operatorname{diag}(\sqrt{H_{ii}})$, then

$$\|\sin\Theta(X_1,\widetilde{X}_1)\|_F \leq \frac{\kappa(A)}{1-4\alpha\,\sqrt{\kappa(A)}}\cdot\frac{2\alpha}{\overline{\operatorname{rg}}(\widetilde{\Lambda}_1,\Lambda_2)}. \tag{37}$$

Proof. The proof is a combination of the proofs of Theorems 4 and 5. As in the proof of Theorem 4, the relations (24), (27) and (28) imply

$$\|\sin\Theta(X_1,\widetilde{X}_1)\|_F \le \|X\|\,\|\widetilde{X}\|\,\|X_2^* J\widetilde{X}_1\|_F. \tag{38}$$

By using the $(2,1)$ block of (31) and the definition (35), we get

$$|X_2^* J\widetilde{X}_1|_{ij} \le (|X_2^*\widehat{E}J\widetilde{X}_1|_{ij} + |X_2^* JE\widetilde{X}_1|_{ij}) \cdot \frac{1}{\overline{\mathrm{rg}}(\widetilde{\Lambda}_1,\Lambda_2)}.$$

Therefore,

$$\|X_2^* J\widetilde{X}_1\|_F \le \frac{\|X_2^*\widehat{E}J\widetilde{X}_1\|_F + \|X_2^* JE\widetilde{X}_1\|_F}{\overline{\mathrm{rg}}(\widetilde{\Lambda}_1,\Lambda_2)} \le \|X\|\,\|\widetilde{X}\| \cdot \frac{2\alpha}{\overline{\mathrm{rg}}(\widetilde{\Lambda}_1,\Lambda_2)}.$$

Now (36) follows by inserting this and (34) into (38). Finally, (37) follows from (36) and Theorem 1. ∎

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